Misspecification in Dynamic Models

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Within the class of ARMAX models we consider the effects omitted variables have on the dynamic shape and the exogeneity properties of economic relations. We provide necessary and sufficient conditions for dynamic shape, Granger causation, linear independence and structural invariance being preserved in the misspecified equation. We show that, barring particular cases, the misspecified equation does not preserve the properties of the "true" relation. In particular, the misspecified equation has a complex dynamic shape even though the true relation is static. These results apply to several misspecification problems, such as measurement errors, non-linearity, aggregation over agents and over time. We argue that misspecification must be regarded as a major source of dynamics for macroeconomic relations.

KEY WORDS: Stationary process, ARMAX model, Orthogonal projection, Dynamic shape, Exogeneity.

INTRODUCTION

The consequences of the omission of a relevant variable in static models are well known. By contrast, little has been said on this subject within a dynamic framework. The issue is extremely interesting, since, as shown in Section 3, a wide class of misspecification problems can be thought of as omitted-variables problems. Examples are non-linearity, errors in variables, unobserved components, signal extraction, aggregation over agents, temporal aggregation.

The questions we deal with in this paper are well described by means of the following example. Assume that the variable $Y_t$ satisfies the static relation

$$Y_t = aX_t + cZ_t + E_t,$$  \hfill (1)

where $aX_t + cZ_t$ is the best linear predictor of $Y_t$, given $X_t$, $Z_t$, and all past values of $Y_t$, $X_t$ and $Z_t$. Assume further that we omit the variable $Z_t$, e.g. because data are not available, and specify the relation as

$$Y_t = \beta X_t + R_t.$$  \hfill (2)

The first question is: does equation (2) provide the best linear predictor of $Y_t$ given all the available information, i.e. $X_t$ and all of the past values of $X_t$ and $Y_t$? Put another way, is there any dynamic equation linking $Y_t$ and $X_t$ whose
prediction-error variance is less than the variance of $R_t$? Consider, for instance, an equation much more general than (2), i.e. the ARMAX

$$\alpha(L)Y_t = \beta(L)X_t + \gamma(L)W_t,$$

(3)

where $\alpha(L)$, $\beta(L)$ and $\gamma(L)$ are polynomials in the lag operator $L$ and $Y_t - W_t$ is the best linear predictor of $Y_t$ given the available information. Is it generally true that equation (3) reduces to the static form (2)? If the answer is negative, what is the dynamic shape of the misspecified equation (3)? Can we state for instance that in general $\alpha(L) = \gamma(L)$, so that the misspecified equation is a rational distributed lag?

The problem can be generalized by allowing for a more general specification of the "true" relation. Assume for instance that $Y_t$ follows the rational distributed lag

$$Y_t = \frac{a(L)}{b(L)} X_t + \frac{c(L)}{d(L)} Z_t + \xi_t.$$

Can we state that the dynamic shape of the true relation is robust with respect to misspecification, i.e. also equation (3) is a rational distributed lag?

The questions above are concerned with the dynamic shape of equation (3). However, other important problems arise, concerning the exogeneity properties and the forecast performance of the misspecified equation.

First, it is easily seen that equation (3) cannot perform better than equation (1) in predicting $Y_t$. Indeed, it can be shown that $W_t = A_t + \xi_t$, where $A_t$ is orthogonal to $\xi_t$; that is, the prediction error of the misspecified equation decomposes into two components: the prediction error of the true relation (1) and an additional error arising from misspecification, which can be termed misspecification error. The question is: under what conditions does this additional error vanish, so that the misspecified equation retains all the information embedded into $Z_t$?

Second, suppose that in equation (1) $\xi_t$ is orthogonal to all future values of the processes $X_t$ and $Z_t$, i.e. $Y_t$ does not Granger cause either $X_t$ or $Z_t$, given the past of both the processes. Is there any feedback in relation (3)?

Third, assume that $Y_t$ does not depend on $X_t$ (given $Z_t$), i.e. in equation (1) $a = 0$ and $\xi_t$ is orthogonal to the future of $X_t$. Under what conditions is $Y_t$ independent of $X_t$ in the misspecified equation?

Last, suppose that the parameters of the true relation are invariant with respect to some policy intervention, so that Lucas’ (1976) critique does not apply to equation (1). Does the invariance property hold for the parameters of equation (3)?

In this paper we provide necessary and sufficient conditions for the properties listed above to be robust with respect to misspecification. The central result is a rather negative one. All of the dynamic properties of economic relations — i.e. prediction-error variance, dynamic shape, unidirectional causation, linear independence, parameter invariance — are destroyed upon misspecification, unless the joint covariance structure of the explanatory variables $X_t$ and $Z_t$
satisfies restrictions which in most cases are unlikely to hold. The misspecified
equation has a complex ARMAX shape, even though the true relation is static.
The dependent variable $Y_t$ Granger causes $X_t$, even though it does not Granger
cause $X_t$ conditionally on $Z_t$. The process $X_t$ enters the misspecified equation,
even though $Y_t$ is independent of $X_t$ in the true relation. Structural invariance
is lost when the relation is misspecified, so that policy analysis may be seriously
misleading, independently of Lucas’ rational expectations argument.

These results suggest two observations. First, we must be very careful in
specifying macroeconomic relations. In Section 3 we show that misspecification
problems arising from aggregation may be mitigated by including “distributive”
variables among the regressors. A similar point is made by Lau (1982), Jorgenson
et al. (1982) and Stoker (1984, 1986), within a static, non-linear set-up.

Second, we must be very careful in interpreting estimated macro relations.
Despite our conscientiousness, macroeconomic relations are likely to be affected
by many misspecification problems, particularly aggregation over agents and over
time, in addition to non-linearity and measurement errors. Therefore, the main
dynamic properties of these relations may well be due to misspecification.

The dynamic shape of macroeconomic relations is usually explained as re-
resulting from individual expectations, adjustment costs or agent’s inertia (see e.g.
Hendry et al. 1984, pp.1037-40). Equations with lagged dependent variables, for
instance, may arise from search costs, transaction costs and optimization costs,
or slow agents’ reactions due to habits and lags in perceiving changes. Moreover,
distributed lags may result from elimination of agents’ expectations. While not
denying the importance of these reasons, our results strongly suggest that lin-
earization and aggregation both over agents and over time are major sources of
the complex dynamic shape of macroeconomic relations.

Some results on the issues addressed here are already known, though they
have never been explored in a systematic way. In particular, there is a large
literature on Granger causation. Tiao and Wei (1978) show that unidirectional
causation is not robust with respect to temporal aggregation. The same con-
clusion holds for unobserved components models (see e.g. Nerlove et al. 1979,
pp.167-168; Sargent 1987, pp.346-348) and aggregation over agents (Lippi 1988a,
1988b). By contrast, little work has been done on the dynamic shape of misspeci-
fied equations. Weiss (1984) discusses the lag length of relations linking temporal
aggregates of time series. Lippi (1988a, 1988b) and Lippi and Forni (1990) show
that aggregation over agents completely modifies the dynamic shape of economic
relations.

The outline of the paper is as follows. Section 1 provides assumptions and
definitions. The main results are presented in Section 2. Section 3 collects ex-
amples and applications to measurement errors, non-linear models, aggregation
over agents, unobserved components and temporal aggregation. In this Section
it is shown that the main results stated in the previous literature can be easily
derived from the proposition proved in Section 2. In Section 4 some concluding
remarks are provided.
1. ASSUMPTIONS AND DEFINITIONS

1.1 The true relation

The time series $Y_t$, $X_t$, and $Z_t$ are zero-mean, jointly covariance stationary, purely non-deterministic processes with non-singular, rational spectral-density matrix. The process $Y_t$ follows the relation

$$Y_t = \frac{a(L)}{b(L)}X_t + \frac{c(L)}{d(L)}Z_t + E_t,$$

(4)

where the functions in the lag operator $L$ are polynomials. Relation (4) is the true relation. In equation (4) the roots of $b(L)$ and $d(L)$ are of modulus greater than one, $b(0) = d(0) = 1$, $a(L)$ and $b(L)$ as well as $c(L)$ and $d(L)$ have no common roots, $E_t$ is a white-noise disturbance orthogonal to $Y_{t-h}$, $h > 0$ and $X_{t-k}$, $Z_{t-k}$, all $k$. These conditions ensure that $[a(L)/b(L)]X_t + [c(L)/d(L)]Z_t$ is the best linear predictor of $Y_t$ within the Hilbert space spanned by the past of $Y_t$ and the past, present and future of $X_t$ and $Z_t$, while $E_t$ is the prediction error.

1.2 The explanatory variables

Since the vector process $(Y_t, Z_t, X_t)$ has a rational spectral-density matrix (i.e. possesses an ARMA representation), the vector process $(Z_t, X_t)$ has a rational spectral-density matrix (Lütkepohl 1984), and admits the representation:

$$e(L)Z_t = f(L)X_t + g(L)U_{zt},$$

(5)

$$k(L)X_t = h(L)Z_t + g(L)U_{xt},$$

(6)

where the polynomials in $L$ satisfy the following conditions:

(i) $g(L)$ has no factors common to all other polynomials;

(ii) $e(0) = k(0) = g(0) = 1$, $h(0) = 0$;

(iii) the roots of $g(L)$ are of modulus greater or equal to one;

(iv) $e(L)k(L) - f(L)h(L)$ vanishes only outside the closed unit circle, except for zeroes of $g(L)$. Moreover,

(v) $U_{zt}$ is orthogonal to $X_{t-k}$, $Z_{t-h}$, $k \geq 0$, $h > 0$, while $U_{xt}$ is orthogonal to $X_{t-h}$, $Z_{t-h}$, $h > 0$.

1 The variables $Z_t$ and $X_t$ have a rational joint Wold representation, i.e.

$$
\begin{pmatrix}
Z_t \\
X_t
\end{pmatrix} = A(L)
\begin{pmatrix}
V_{zt} \\
V_{xt}
\end{pmatrix},
$$

(7)

where $A(L)$ is a matrix of rational functions in $L$ with no poles of modulus less or equal to one, $A(0) = I$, $\text{det}[A(L)]$ has no roots of modulus less than one and $(V_{zt}, V_{xt})$ is a vector white noise orthogonal to $Z_{t-h}$, $X_{t-h}$, $h > 0$. Representation (5)-(6) is obtained from the Wold representation by premultiplying both sides of (7) by

$$
A^*(L)
\begin{pmatrix}
1 \\
-\frac{\text{cov}(V_{zt}, V_{xt})}{\text{var}(V_{xt})}
\end{pmatrix},
$$

where $A^*(L)$ is the adjoint of $A(L)$, and by eliminating denominators and common factors.
By construction, $U_{Z_t}$ and $U_{X_t}$ are white noises orthogonal at all leads and lags. $U_{Z_t}$ is the residual of the orthogonal projection of $Z_t$ on the space spanned by $X_t$ and the lagged variables $X_{t-k}, Z_{t-h}$, $k \geq 0, h > 0$, while $U_{X_t}$ is the residual of the orthogonal projection of $X_t$ on the space spanned by $X_{t-h}, Z_{t-h}$, $h > 0$. Therefore, $Z_t - U_{Z_t}$ is the best linear predictor of $Z_t$ within the first space, while $X_t - U_{X_t}$ is the best linear predictor of $X_t$ within the second one. We will refer to equation (5) as the model of $Z_t$ conditional to $X_t$ or simply the relation linking $Z_t$ and $X_t$. Equation (6) is the marginal model of $X_t$.

It will prove useful to define the set of admissible parameters of equations (5) and (6). A particular set of values of the parameters of equations (5) and (6), i.e. $\epsilon(L), f(L), g(L), k(L), h(L), \operatorname{var}(U_{Z_t})$ and $\operatorname{var}(U_{X_t})$, is admissible if it satisfies the conditions (i) through (iv) listed above along with $\operatorname{var}(U_{Z_t}) \geq 0$ and $\operatorname{var}(U_{X_t}) \geq 0$. The class of all admissible $\mathcal{F}$ is denoted by $\Phi$. It can be proved that, since the spectral-density matrix of $(Z_t, X_t)$ is non-singular almost everywhere on the interval $[-\pi, \pi]$, i.e. neither $U_{Z_t}$ nor $U_{X_t}$ are zero, representation (5) is unique; that is, given the joint covariance structure of the processes $Z_t$ and $X_t$, the parameters $\mathcal{F}$ listed above are univocally determined by conditions (i) through (v) (see Hannan 1970, chs. 2,3 or Rozanov 1967, chs. 1,2).

The conditions imposed on $E_t$ in equation (4) ensure that the residuals $U_{Z_t}$ and $U_{X_t}$ in equation (5) are orthogonal not only to the past of the processes $X_t$ and $Z_t$ but also to the past of the process $Y_t$. In fact, $U_{Z_t}$ and $U_{X_t}$ are orthogonal to the whole process $E_t$, while the lagged variables $Y_{t-h}, h > 0$, are linear combinations of the past of $X_t, Z_t$ and $E_t$. Therefore $Y_t$ does not Granger cause either $X_t$ or $Z_t$, given the past of both the processes.

1.3 The misspecified equation

The misspecified model is

\begin{align*}
\alpha(L)Y_t &= \beta(L)X_t + \gamma(L)W_t \quad (8) \\
\delta(L)X_t &= \theta(L)Y_t + \gamma(L)U_t \quad (9)
\end{align*}

The misspecified relation (8) is the model of $Y_t$ conditional to $X_t$: the polynomials in (8) and the process $W_t$ satisfy the restrictions imposed on the corresponding polynomials and on the process $U_{Z_t}$ in equation (5). Therefore, $Y_t - W_t$ is the best linear predictor of $Y_t$ given $X_t$ and all past values of $X_t$ and $Y_t$. Equation (9) is the corresponding marginal model; $X_t - U_t$ is the best linear predictor of $X_t$ given all past values of $X_t$ and $Y_t$.

A particular choice of the parameters in equations (8) and (9), i.e. $\alpha(L), \beta(L), \gamma(L), \delta(L), \theta(L), \operatorname{var}(W_t)$ and $\operatorname{var}(U_t)$, is denoted by $\mathcal{O}$, while $\Omega$ indicates the set of all admissible $\mathcal{O}$. The spectral-density matrix of the vector $(Y_t, X_t)$ is denoted by

$$S = \begin{pmatrix} S_{YY} & S_{YX} \\ S_{XY} & S_{XX} \end{pmatrix}$$
The assumptions in 1.1 ensure that $S$ is a matrix of rational functions in $e^{-\lambda t}$, $-\pi \leq \lambda < \pi$. Moreover, $S$ is non-singular almost everywhere. Therefore representation (8)-(9) always exists and is unique.

1.4 The dynamic shape

We say that the misspecified equation (8) is
(a) rational distributed lag (RDL), if and only if $\alpha(L) = \gamma(L)$ and the roots of $\gamma(L)$ are of modulus greater than one;
(b) unrestricted ARMAX, if and only if it is not a rational distributed lag. Moreover, if the misspecified equation is RDL, it is
(a') finite distributed lag (FDL), if and only if $\gamma(L) = 1$;
(a'') static, if and only if it is FDL and $\beta(L) = \beta$.

Similar definitions hold for equation (5). By definition, the true relation (4) cannot take the shape (b).

1.5 Causation, independence and invariance

There is a one-way causation in the misspecified model, that is $Y_t$ does not Granger cause $X_t$, if and only if $\theta(L) = 0$ in equation (9), i.e. $U_t$ is the residual of the orthogonal projection of $X_t$ on its own past. There is a one-way causation in model (5)-(6) if and only if $h(L) = 0$. There is always a one-way causation in the true model, since by construction $Y_t$ does not Granger cause either $X_t$ or $Z_t$ conditionally on both the processes.

$Y_t$ is independent of $X_t$ if, and only if, $\beta(L) = 0$ in equation (8) and $Y_t$ does not Granger cause $X_t$ (i.e. $Y_t$ and $X_t$ are orthogonal at all leads and lags). $Y_t$ is independent of $X_t$ conditionally on $Z_t$ if and only if $\alpha(L) = 0$ in relation (4).

The misspecified relation (8) is invariant with respect to $\Phi$ if and only if the parameters of (8), i.e. $\alpha(L), \beta(L), \gamma(L)$ and var($W_t$), do not depend on the parameters in $\Phi$ — that is, they do not depend on the joint covariance structure of $Z_t$ and $X_t$. A similar definition holds for the true relation.

1.4 The misspecification error

Consider the projection of $[c(L)/d(L)]Z_t$ on $X_t$ and the past values of $X_t$ and $Y_t$. Call this projection $P_t$ and the residual $A_t$. Then

$$Y_t = \frac{a(L)}{b(L)} X_t + P_t + A_t + E_t.$$ 

It turns out that $A_t + E_t = W_t$, where $W_t$ is the residual of the misspecified relation. In fact, both $A_t$ and $E_t$ are orthogonal to $X_t$ and the past values of $X_t$ and $Y_t$, while $[a(L)/b(L)]X_t + P_t$ belongs to the Hilbert space spanned by the same variables. Moreover, $E_t$ is orthogonal to $P_t$ and $Z_{t-k}, k \geq 0$; since $A_t = [c(L)/d(L)]Z_t - P_t$, $E_t$ is orthogonal to $A_t$, so that var($W_t$) = var($A_t$) + var($E_t$).

Hence, the prediction error $W_t$ decomposes into two orthogonal components, $E_t$, that is the prediction error of the true relation, and the additional error...
\( A_t \). The variance of \( A_t \) measures the information we lost when predicting \( Y_t \) using the misspecified relation instead of the true relation. The process \( A_t \) is the misspecification error; i.e. the component of the prediction error arising from misspecification.

2. RESULTS

2.1 Prediction

**Proposition 1.** The misspecification error \( A_t \) is zero if and only if \( c(L) = 0 \).

**Proof.** The sufficiency part is obvious. To prove necessity, assume \( W_t = E_t \).

From (4) and (8) it is obtained

\[
\alpha(L) \frac{c(L)}{d(L)} Z_t = \left[ \beta(L) - \frac{a(L)}{b(L)} \alpha(L) \right] X_t + [\gamma(L) - \alpha(L)] E_t. \quad (10)
\]

The last term on the right-hand side belongs to the space spanned by \( E_{t-h}, \ h \geq 0 \). However, it belongs also to the space spanned by the present and past of the processes \( Z_t \) and \( X_t \). Since the former space is orthogonal to the latter, the last term of (10) is orthogonal to itself and therefore is zero. If \( c(L) \neq 0 \), equation (10) implies that the processes \( Z_t \) and \( X_t \) span the same space and have a singular spectral-density matrix, contrary to the assumptions in 1.1.

2.2 Invariance

**Lemma 1.** Define \( \Omega' \) the set of all parameter choices \( \mathcal{O} \) of the misspecified relation such that

\[
\text{var}(E_t) \leq S_{YY} - \frac{S_{XY} S_{YX}}{S_{XX}} \quad (11)
\]

on the interval \([ -\pi, \pi ] \). If \( c(L) \neq 0 \), whatever \( \mathcal{O} \) in \( \Omega' \) can be obtained for the misspecified equation by a suitable choice of \( \mathcal{F} \) in \( \Phi \), i.e. by a suitable choice of the parameters of equations (5) and (6).

**Proof.** Take some \( \mathcal{O}^* \) in \( \Omega' \). This determines univocally \( S = S^* \). Define

\[
R^* = S^* - \begin{pmatrix} \text{var}(E_t) & 0 \\ 0 & 0 \end{pmatrix}
\]

and

\[
G^* = \begin{pmatrix} d(z) \\ c(z) \\ 0 \end{pmatrix} - \frac{a(z)}{b(z)c(z)} \begin{pmatrix} d(z^{-1}) \\ c(z^{-1}) \\ 0 \end{pmatrix} R^* \begin{pmatrix} d(z^{-1}) \\ c(z^{-1}) \\ 0 \end{pmatrix} - \frac{a(z^{-1})}{b(z^{-1})c(z^{-1})} \begin{pmatrix} d(z^{-1}) \\ c(z^{-1}) \\ 1 \end{pmatrix}, \quad (12)
\]
where \( z = e^{-i\lambda}, -\pi \leq \lambda < \pi \). Condition (11) along with \( c(z) \neq 0 \) ensure that matrices \( R^* \) and \( G^* \) are well-defined rational spectral-density matrices. Therefore there is always one (and only one) \( \mathcal{F}^* \) in \( \Phi \) such that the spectral-density matrix of \( (Z_t, X_t) \) is equal to \( G^* \). If we choose \( \mathcal{F}^* \) for the parameters of equations (5) and (6) we obtain \( S = S^* \) by equation (12). This determines univocally \( \mathcal{O} = \mathcal{O}^* \) for the parameters of the misspecified model.

**Proposition 2.** Assume that the true relation is invariant with respect to \( \Phi \). Then, the misspecified equation is invariant with respect to \( \Phi \) if and only if \( c(L) = 0 \).

**Proof.** If \( c(L) = 0 \) the misspecified equation and the true equation are the same. If \( c(L) \neq 0 \), take two choices \( \mathcal{O}^* \) and \( \mathcal{O}^{**} \) in \( \Omega' \) such that at least one of the equalities \( \alpha^*(L) = \alpha^{**}(L), \beta^*(L) = \beta^{**}(L), \gamma^*(L) = \gamma^{**}(L), \var(W_t)^* = \var(W_t)^{**} \) does not hold. By Lemma 1, both \( \mathcal{O}^* \) and \( \mathcal{O}^{**} \) can be obtained for the misspecified equation by varying \( \mathcal{F} \) in \( \Phi \).

2.3 Granger causation

**Lemma 2.** If \( c(L) \) does not vanish within the unit circle, the model of the omitted term \( S_t = [c(L)/d(L)]Z_t \) conditional on \( X_t \) is

\[
e(L)d(L)S_t = f(L)c(L)X_t + \frac{c(L)g(L)}{c(0)}[c(0)U_{Zt}], \tag{13}
\]

while the corresponding marginal model is

\[
\frac{k(L)c(L)}{c(0)}X_t = \frac{h(L)d(L)}{c(0)}S_t + \frac{c(L)g(L)}{c(0)}U_{Xt} \tag{14}.
\]

**Proof.** Equations (13) and (14) are obtained from equations (5) and (6). Since the variables \( S_{t-h}, h > 0, \) belong to the space spanned by \( Z_{t-h}, h > 0, \) it follows that \( c(0)U_{Zt} \) is orthogonal to \( S_{t-h}, X_{t-h}, h > 0, k \geq 0, \) while \( U_{Xt} \) is orthogonal to \( S_{t-h}, X_{t-h}, h > 0. \) Moreover, \( c(L)g(L) \) does not vanish within the unit circle because of the assumptions on \( c(L) \). The other conditions on the polynomials in \( L \) are easily verified.

**Proposition 3.** \( Y_t \) does not Granger cause \( X_t \) if and only if the omitted term \( S_t = [c(L)/d(L)]Z_t \) does not Granger cause \( X_t \).

**Proof.** Call \( Q_t \) the orthogonal projection of \( Y_t \) on the space \( \mathcal{H}_X \) spanned by \( X_{t-k}, \) all \( k, \) and \( H_t \) the residual. Hence, \( S_t = \Lambda_t + K_t, \) where \( \Lambda_t = Q_t - [a(L)/b(L)]X_t \) and \( K_t = H_t - E_t. \) Since both \( H_t \) and \( E_t \) are orthogonal to \( \mathcal{H}_X, K_t \) is orthogonal to \( \mathcal{H}_X. \) On the other hand, \( \Lambda_t \) belongs to \( \mathcal{H}_X; \) therefore, \( \Lambda_t \) is the projection of \( S_t \) on \( \mathcal{H}_X. \) It is clear from the definition of \( \Lambda_t \) that \( \Lambda_t \) belongs to the subspace of \( \mathcal{H}_X \) spanned by the present and past of \( X_t \) if, and only if, \( Q_t \) belongs to this subspace.
Proposition 4. (a) If $Z_t$ does not Granger cause $X_t$, then $Y_t$ does not Granger cause $X_t$. (b) If $c(L)$ does not vanish within the unit circle, $Y_t$ does not Granger cause $X_t$ if and only if $Z_t$ does not Granger cause $X_t$.

Proof. (a) The projection of $S_t$ on the whole process $X_t$ is $A_t = [c(L)/d(L)]\Pi_t$, where $\Pi_t$ is the projection of $Z_t$ on the same space. If $\Pi_t$ belongs to the subspace spanned by $X_{t-k}$, $k > 0$, then also $A_t$ belongs to this subspace, so that $S_t$ does not Granger cause $X_t$ and by Proposition 3 $Y_t$ does not Granger cause $X_t$. (b) It is clear from Lemma 2, equation (14), that if $c(L)$ does not vanish within the unit circle, then $S_t$ Granger causes $X_t$ if and only if $Z_t$ Granger causes $X_t$. The result follows from Proposition 3.

2.4 Independence

Proposition 5. If $Y_t$ is independent of $X_t$ conditionally on $Z_t$, $Y_t$ is independent of $X_t$ if and only if (i) $c(L) = 0$, or (ii) $Z_t$ is independent of $X_t$.

Proof. If $a(L) = 0$, then $S_t = S_{X_t}c(z)/d(z)$, where $z = e^{-i\lambda}$, $-\pi \leq \lambda < \pi$. The right-hand side vanishes if and only if either (i) or (ii) hold.

2.5 Dynamic shape

Proposition 6. If $c(L) \neq 0$, an arbitrary dynamic shape can be obtained for the misspecified equation and the marginal model (9) whatever the dynamic shape of the true relation (4), by suitably setting $\mathcal{F}$ and $\text{var}(E_t)$.

Proof. Take some $\Omega^*$ in $\Omega$ and set $\text{var}(E_t)$ such that (11) is satisfied. The result follows from Lemma 1.

Remark. Assume that the true equation is static. If $\text{var}(E_t) = 0$ then $\Omega = \Omega'$, where $\Omega'$ as in Lemma 1. By Proposition 6, the misspecified equation can take whatever dynamic shape, depending on the parameters in (5) and (6). On the contrary, if $\text{var}(E_t) > 0$ there are some parameters $\Omega$ and some associated spectra $S$ which do not satisfy (11), i.e. $\Omega' \subset \Omega$. The misspecified models characterized by such parameters cannot be obtained from a static equation. The greater $\text{var}(E_t)$, the smaller $\Omega'$ and the larger is the set of these models.

Proposition 7. The misspecified equation is a rational distributed lag if and only if the model of the omitted term $S_t = [c(L)/d(L)]Z_t$ conditional on $X_t$ is a rational distributed lag.

Proof. Assume that in the misspecified equation $\alpha(L) = \gamma(L)$ and $\gamma(L)$ vanishes only outside the closed unit circle. Equation (4) implies

$$S_t = [\beta(L)/\gamma(L) - a(L)/b(L)]X_t + (W_t - E_t).$$

(15)

$W_t$ belongs to the space spanned by $E_t$, $S_t$ and $X_{t-k}$, $k \geq 0$, so that $E_{t-h}$ is orthogonal to $W_t$ for $h > 0$. Since $W_t$ is orthogonal to $Y_{t-h}$, $X_{t-k}$, $h > 0$, $k \geq 0$,
it is also orthogonal to $S_{t-h} = Y_{t-h} - [a(L)/b(L)]X_{t-h} - E_{t-h}$, $h > 0$. Hence $W_t - E_t$ is orthogonal to $S_{t-h}, X_{t-k}, k \geq 0, h > 0$. The model of $S_t$ conditional on $X_t$ is then obtained by (15) eliminating denominators and common factors, so that it is RDL. Conversely, let us assume that the model of $S_t$ conditional on $X_t$ is

$$S_t = [f'(L)/g'(L)]X_t + U_{St},$$

(16)

Then

$$Y_t = [a(L)/b(L) + f'(L)/g'(L)]X_t + (E_t + U_{St}).$$

(17)

$E_t$ and $U_{St}$ are orthogonal at all leads and lags, since $E_t$ is orthogonal to the processes $S_t$ and $X_t$. Moreover, $U_{St}$ is orthogonal by definition to $S_{t-h}, X_{t-k}, h > 0, k \geq 0$. Therefore $U_{St}$ is orthogonal to $Y_{t-h} = [a(L)/b(L)]X_{t-h} + S_{t-h} + E_{t-h}, h > 0$. Hence, $E_t + U_{St}$ is orthogonal to $X_{t-k}, Y_{t-h}, k \geq 0, h > 0$. Thus, the misspecified equation is obtained from (17) eliminating denominators and common factors, so that it is RDL.

Proposition 8. If $c(L)$ does not vanish within the unit circle, the misspecified equation is a rational distributed lag if and only if$^2 c(L)d(L) = c(L)g(L)/c(0)$. In this case, $\alpha(L) = \gamma(L) = b(L)g(L), \beta(L) = a(L)g(L) + c(0)f(L)b(L)$ and $W_t = c(0)U_{St} + E_t$.

Proof. By Lemma 2, if $c(L)$ does not vanish within the unit circle, equation (13) is the model of $S_t$ conditional on $X_t$. This model is RDL if and only if $c(L)d(L) = c(L)g(L)/c(0)$, so that by Proposition 7 the last condition is equivalent to the misspecified equation being RDL. The expressions for $\alpha(L), \beta(L), \gamma(L)$ and $W_t$ follow from equations (13), (16) and (17).

Proposition 9. If the true relation is static and $c(0) \neq 0$, the misspecified equation is (i) RDL, (ii) FDL, (iii) static, if and only if the relation linking $Z_t$ and $X_t$ is respectively (i) RDL, (ii) FDL, (iii) static.

Proof. By Proposition 8, if $c(L) = c(0) \neq 0, a(L) = a(0)$ and $d(L) = b(L) = 1,$ the misspecified equation is RDL if and only if $c(L) = g(L)$, i.e. the model of $Z_t$ conditional on $X_t$ is RDL. Moreover, in this case $\alpha(L) = \gamma(L) = c(L) = \gamma(L)$ and $\beta(L) = a(0)e(L) + c(0)f(L)$. Therefore $\alpha(L) = \gamma(L) = 1$, i.e. the misspecified equation is FDL, if and only if $c(L) = g(L) = 1$, i.e. the model of $Z_t$ conditional on $X_t$ is FDL. In this case, $\beta(L) = a(0) - c(0)f(L)$. It follows that $\beta(L) = \beta(0)$ (the misspecified equation is static) if and only if $f(L) = f(0)$ (the model of $Z_t$ conditional on $X_t$ is static).

$^2$ This condition implies that (a) if the true relation is static in $Z_t$, i.e. $c(L) = c(0)$ and $d(L) = 1$, the misspecified equation is RDL if and only if the model of $Z_t$ conditional on $X_t$ is RDL; (b) if $c(L) = 0$ and the model of $Z_t$ conditional on $X_t$ is RDL, then the misspecified relation is RDL if and only if the true relation is static in $Z_t$. 

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Proposition 10. If \( Z_t \) does not Granger cause \( X_t \), then in equations (8) and (9)

\[
\alpha(L) = b(L)d(L)c(L) \\
\beta(L) = a(L)d(L)e(L) + c(L)f(L)b(L) \\
\gamma(L)/\delta(L) = k(L)/g(L)
\] (18)

and \( \gamma(L) \), \( \text{var}(W_t) \) are such that

\[
|\gamma(z)|^2 \text{var}(W_t) = |b(z)|^2 (|c(z)g(z)|^2 \text{var}(U_{Z_t}) + |d(z)e(z)|^2 \text{var}(E_t)),
\] (19)

where \( z = e^{-i\lambda}, -\pi < \lambda \leq \pi \).

Proof. From equations (4), (5) and (6) we obtain

\[
S_{XX} = \frac{|g(z)|^2}{|k(z)|^2} \text{var}(U_{X_t})
\]

\[
S_{YX} = \frac{|g(z)|^2}{|k(z)|^2} \text{var}(U_{X_t}) \left( \frac{a(z)}{b(z)} - \frac{c(z)f(z)}{d(z)e(z)} \right)
\]

\[
S_{YY} = \frac{|g(z)|^2}{|k(z)|^2} \text{var}(U_{X_t}) \left( \frac{a(z)}{b(z)} - \frac{c(z)f(z)}{d(z)e(z)} \right)^2 + \frac{|c(z)g(z)|^2}{|d(z)e(z)|^2} \text{var}(U_{Z_t}) + \text{var}(E_t).
\]

Imposing the equalities (18) and (19) the same spectra are obtained from (8) and (9).

3. EXAMPLES AND APPLICATIONS

3.1 Errors in variables

Assume that the variables \( \hat{Y}_t \) and \( Z_t \) are linked by the relation

\[
\hat{Y}_t = \frac{c(L)}{d(L)} Z_t + R_t,
\]

where \( c(L) \) and \( d(L) \) do not vanish within the unit circle and \( R_t \) is orthogonal to \( \hat{Y}_{t-h}, Z_{t-k}, h > 0, \) all \( k \). Assume further that \( \hat{Y}_t \) and \( Z_t \) are subject to a measurement error. The observed values are \( Y_t = \hat{Y}_t + O_t \) and \( X_t = Z_t + U_t \), where the errors \( O_t \) and \( U_t \) are joint white noises with diagonal variance-covariance matrix, orthogonal to \( \hat{Y}_{t-k} \) and \( Z_{t-k} \) for all \( k \). Hence

\[
Y_t = \frac{c(L)}{d(L)} Z_t + E_t,
\] (20)
where \( E_t = \delta_t + R_t \). It is easily verified that \( E_t \) is orthogonal to \( Y_{t-h}, X_{t-k}, Z_{t-k}, h > 0, k, \) so that (20) has the same properties as equation (4).

Assume that the univariate Wold representation of \( Z_t \) is \( Z_t = [\mu(L)/\nu(L)]V_{Zt} \) where \( \mu(L) \) has no unit-modulus roots. It follows that

\[
\begin{pmatrix}
    Z_t \\
    X_t
\end{pmatrix}
= \begin{pmatrix}
    \mu(L)/\nu(L) & 0 \\
    \mu(L)/\nu(L) & 1
\end{pmatrix}
\begin{pmatrix}
    V_{Zt} \\
    U_t
\end{pmatrix}.
\tag{21}
\]

The joint Wold representation of \( Z_t \) and \( X_t \) is obtained from (21) by inserting the factor

\[
I = \begin{pmatrix}
    1 & 0 \\
    -1 & 1
\end{pmatrix}
\begin{pmatrix}
    1 & 0 \\
    1 & 1
\end{pmatrix}
\]

between the matrix and the vector on the right-hand side. Following the steps indicated in Subsection 1.2, footnote 1, we get

\[
\begin{pmatrix}
    (1-p)\nu + p\mu \\
    \nu - \mu
\end{pmatrix}
\begin{pmatrix}
    Z_t \\
    X_t
\end{pmatrix}
= \mu
\begin{pmatrix}
    U_{Zt} \\
    U_{Xt}
\end{pmatrix}.
\tag{22}
\]

where \( p = \text{var}(V_{Zt})/[(\text{var}(V_{Zt}) + \text{var}(U_t))] \), \( U_{Zt} = (1-p)V_{Zt} - pU_t \) and \( U_{Xt} = U_t + V_{Zt} \). The first line of (22) is the model of \( Z_t \) conditional on \( X_t \), while the second line is the marginal model of \( X_t \).

Consider now the misspecified model, that is the model of \( Y_t \) conditional on \( X_t \) and the corresponding marginal model. By Propositions 1, the misspecification error is zero if and only if \( \sigma(L) = 0 \), i.e. \( Y_t \) is a white-noise process orthogonal to \( Z_t \) at all leads and lags. This condition is also equivalent to the misspecified relation being invariant with respect to the autocovariance structure of \( Z_t \). Since \( \sigma(L) \) does not vanish within the unit circle, by Proposition 4 \( Y_t \) does not Granger cause \( X_t \) if and only if \( \mu(L) = \nu(L) \), i.e. \( Z_t \) is white noise. Lastly, by Proposition 8 the misspecified relation is RDL if and only if

\[
d(L)[(1-p)\nu(L) + p\mu(L)] = c(L)\mu(L)/c(0).
\]

It follows that if the true relation is static, either \( \nu(L) = \mu(L) \), i.e. \( Z_t \) is white noise, so that the misspecified equation is static, or \( Z_t \) is not white noise, in which case the misspecified relation is an unrestricted ARMAX.

3.2 Non-linearity

In this Subsection we assume that \( Y_t \) and \( X_t \) are strictly stationary and are linked by the static quadratic relation

\[
Y_t = aX_t + c[X_t^2 - \text{var}(X_t)] + E_t,
\]

where \( E_t \) is independent of \( Y_{t-h}, X_{t-k}, h > 0, \) all \( k \). The problem is to find the linear model of \( Y_t \) and \( X_t \).
To simplify matters, assume that the explanatory variable $X_t$ is AR(1), i.e. $X_t = \phi X_{t-1} + U_t$, where $U_t$ is independent of $X_{t-h}$, $h > 0$. It follows that $Z_t = X_t^2 - \text{var}(X_t)$ satisfies

$$Z_t = \phi^2 Z_{t-1} + R_t,$$

where $R_t = U_t^2 - \text{var}(U_t) + 2\phi U_t X_{t-1}$. It is easily seen that $R_t$ is a zero-mean white noise independent of $X_{t-h}$ and $Z_{t-h}$ for $h > 0$. Therefore $Z_t$ and $X_t$ are jointly covariance stationary and their Wold representation is

$$
\begin{pmatrix}
Z_t \\
X_t
\end{pmatrix}
= 
\begin{pmatrix}
1/(1-\phi^2 L) & 0 \\
0 & 1/(1-\phi L)
\end{pmatrix}
\begin{pmatrix}
R_t \\
U_t
\end{pmatrix}.
$$

The model of $Z_t$ and $X_t$ is

$$
\begin{pmatrix}
1-\phi^2 L & -p(1-\phi L) \\
0 & 1-\phi L
\end{pmatrix}
\begin{pmatrix}
Z_t \\
X_t
\end{pmatrix}
= 
\begin{pmatrix}
U_{zt} \\
U_t
\end{pmatrix},
$$

(23)

where $p = E(U_t^3)/E(U_t^2)$ and $U_{zt} = R_t - pU_t$.

Since $Z_t$ does not Granger cause $X_t$, by Proposition 4 $Y_t$ does not Granger cause $X_t$ and, by Proposition 10, the misspecified equation is

$$
(1-\phi^2 L)Y_t = [a(1-\phi^2 L) + cp(1-\phi L)]X_t + \gamma(L)W_t,
$$

(24)

where $\text{var}(W_t) = c \text{var}(R_t) + \text{var}(E_t)$. The polynomial $\gamma(L)$ is identified by the condition on the roots and by the equality

$$
|\gamma(\xi)|^2 = \frac{c \text{var}(R_t) + \text{var}(E_t)}{c \text{var}(R_t) + \text{var}(E_t)} |1-\phi^2 L|^2,
$$

where $\xi = e^{-i\lambda}$, $-\pi < \lambda \leq \pi$. It is apparent from (24) that a change in the model generating $X_t$ modifies the parameters of the misspecified relation. Moreover, if $X_t$ is white noise, that is $\phi = 0$, the misspecified equation is static; conversely, if $\phi \neq 0$, the misspecified equation is a general ARMAX.

3.3 Aggregation over agents

In this Subsection we analyze a simplified version of Lippi’s (1989) aggregation problem. There are two groups of consumers. Within the first group each consumer follows the static behavioral rule $y_{it} = a_1 x_{it} + e_{it}$, where $y_{it}$ and $x_{it}$ are respectively consumption and income of agent $i$. Within the second group the behavioural rule is $y_{jt} = a_2 x_{jt} + e_{jt}$. Summing over individuals we get, with an obvious notation,

$$
Y_{1t} = a_1 X_{1t} + E_t
$$

(25)

for the first group and

$$
Y_{2t} = a_2 X_{2t} + E_t
$$

(26)
for the second one. We assume that both $E_{1t}$ and $E_{2t}$ are orthogonal to $Y_{1(t-h)}$, $Y_{2(t-h)}$, $X_{1(t-k)}$, $X_{2(t-k)}$, $h > 0$, all $k$. Our aim is to study the model of the aggregate consumption $Y_t = Y_{1t} + Y_{2t}$ conditional on the aggregate income $X_t = X_{1t} + X_{2t}$.

Summing equations (25) and (26) it is obtained

$$Y_t = Z_t + E_t,$$

(27)

where $Z_t = a_1 X_{1t} + a_2 X_{2t}$ and $E_t = E_{1t} + E_{2t}$. It is easily verified that equation (27) satisfies the properties of equation (4), so that in our terminology equation (27) is the true relation. Aggregate consumption depends on the joint distribution (among individuals) of income and propensity to consume. Indeed, the variable $Z_t$ is the covariance of this distribution. If $Z_t$ is not observable, and we substitute the aggregate income $X_t$ for $Z_t$, the resulting equation is misspecified. As we will see in the following Subsection, the misspecified aggregate relation is in general an unrestricted ARMAX.

A similar problem arises if the parameters of the micro equations are all equal, but the micro equations either are non-linear or include unobserved explanatory variables. Consider for instance the micro equations $y_{it} = ax_{it} + c[x_{it}^2 - \text{var}(x_{it})] + e_{it}$. Summing over individuals yields

$$Y_t = aX_t + cZ_t + E_t,$$

where $Z_t = \sum_i x_{it}^2 - \sum_i \text{var}(x_{it})$. Also in this case aggregate consumption does not depend on aggregate income only, but on the variable $Z_t$, which is the variance of the distribution of income minus its mathematical expectation. Therefore $Z_t$ should be included among the explanatory variables in the aggregate relation. However, if data on $Z_t$ are not available, the best we can do is to specify a model linking $Y_t$ and $X_t$. As stated in Proposition 2, the properties of this model depend on the joint covariogram of $Z_t$ and $X_t$.

3.4 Unobserved components

In this Subsection we discuss the unobserved component model of Nerlove et al. (1979, pp.167-68). Since this model is formally identical to the two-groups aggregation model discussed above, the conclusions of this Subsection apply to the aggregation problem as well.

Assume that there are two economic series, $Y_t$ and $X_t$, each one having two components, seasonal and non-seasonal:

$$Y_t = Y_{Nt} + Y_{St}; \quad X_t = X_{Nt} + X_{St}.$$

The non-seasonal components are linked each other by the relation

$$Y_{Nt} = a_N X_{Nt} + E_{Nt},$$

(28)
while the seasonal components follow the relation

\[ Y_{St} = a_s X_{St} + E_{St}. \]  

(29)

The residuals \( E_{Nt} \) and \( E_{St} \) are orthogonal to \( Y_{N(t-h)}, Y_{S(t-h)}, X_{N(t-h)}, X_{S(t-h)}, h > 0, \) all \( k \). The extension to the case of (28) and (29) being RDL is straightforward.

As usual, we are interested in the model of \( Y_t \) conditional to \( X_t \). Summing (28) and (29) gives

\[ Y_t = Z_t + E_t, \]

where \( Z_t = a_{Nt} X_{Nt} + a_{St} X_{St} \) and \( E_t = E_{Nt} + E_{St} \). Clearly, if \( a_N = a_S = a \), then \( Z_t = a X_t \) and no misspecification problems arise. Conversely, if \( a_N \neq a_S \) the spectral-density matrix of \( (Z_t, X_t) \) is non-singular. Since \( c(L) = 1 \), by Proposition 1 the misspecification error is non-zero and by Proposition 2 the model of \( Y_t \) conditional to \( X_t \) is not invariant with respect to the joint model of \( Z_t \) and \( X_t \). In order to obtain this model, we must explore the covariance structure of the explanatory variables \( X_{Nt} \) and \( X_{St} \).

To simplify calculations, we retain Nerlove’s assumption that \( X_{Nt} \) and \( X_{St} \) are orthogonal at all leads and lags. It should be noted, however, that this restriction can be easily dropped in our framework, whereas it is essential in Nerlove’s one. The joint Wold representation of \( X_{Nt} \) and \( X_{St} \) is therefore

\[
\begin{pmatrix}
X_{St} \\
X_{Nt}
\end{pmatrix} = 
\begin{pmatrix}
\mu(L)/\nu(L) & 0 \\
0 & \phi(L)/\psi(L)
\end{pmatrix}
\begin{pmatrix}
V_{St} \\
V_{Nt}
\end{pmatrix}.
\]

It follows that

\[
\begin{pmatrix}
Z_t \\
X_t
\end{pmatrix} = 
\begin{pmatrix}
a_S \mu/\nu & a_N \phi/\psi \\
\mu/\nu & \phi/\psi
\end{pmatrix}
\begin{pmatrix}
V_{St} \\
V_{Nt}
\end{pmatrix}.
\]

Inserting the factor

\[
\frac{1}{a_S - a_N}
\begin{pmatrix}
1 & -a_N \\
-a_N & a_S
\end{pmatrix}
\begin{pmatrix}
a_S & a_N \\
a_S & 1
\end{pmatrix}
\]

between the matrix and the vector on the right-hand side, we obtain the joint Wold representation of \( Z_t \) and \( X_t \), i.e.

\[
\begin{pmatrix}
Z_t \\
X_t
\end{pmatrix} = 
\begin{pmatrix}
1 & -a_N \\
a_S - a_N & a_S
\end{pmatrix}
\begin{pmatrix}
a_S \mu/\nu - a_N \phi/\psi & a_S a_N (\phi/\psi - \mu/\nu) \\
\mu/\nu - \phi/\psi & a_S \phi/\psi - a_N \mu/\nu
\end{pmatrix}
\begin{pmatrix}
V_{Zt} \\
V_{Xt}
\end{pmatrix},
\]

where \( V_{Zt} = a_S V_{St} + a_N V_{Nt} \) and \( V_{Xt} = V_{St} + V_{Nt} \). The model of \( Z_t \) conditional on \( X_t \) and the corresponding marginal model are

\[
\begin{pmatrix}
\nu \phi (a_S - p) + \mu \psi (p - a_N) \\
\nu \phi a_N (a_S - p) - \mu \psi a_S (p - a_N)
\end{pmatrix}
\begin{pmatrix}
Z_t \\
X_t
\end{pmatrix} = \mu \phi
\begin{pmatrix}
U_{Zt} \\
U_{Xt}
\end{pmatrix},
\]

(30)
where \( p = \text{cov}(V_{Xt}, V_{Xt}) / \text{var}(V_{Xt}) \), \( U_{Zt} = V_{Zt} - pV_{Xt} \) and \( U_{Xt} = V_{Xt} \).

It is easily seen from (30) that \( Y_t \) does not Granger cause \( X_t \) if and only if \( \nu(L) \phi(L) = \mu(L) \psi(L) \), i.e. the autocorrelation structures of the seasonal and non-seasonal components are the same (Proposition 3). This is Nerlove’s result. Moreover, by Proposition 9, the model of \( Y_t \) conditional on \( X_t \) is static if and only if

\[
\nu \phi a_N (a_S - p)/p + \mu \psi a_S (a_N - p)/p \\
= \nu \phi(a_S - p) + \mu \psi(a_N - p) = \mu \psi(a_S - a_N)
\]

From the first inequality we get \( \nu(L) \phi(L) = \mu(L) \psi(L) \); substituting into the second yields \( \psi(L) = \phi(L) \) and \( \mu(L) = \nu(L) \). Therefore the relation linking \( Y_t \) and \( X_t \) is static if and only if the unobserved components \( X_{Nt} \) and \( X_{St} \) are white noises.

3.5 Temporal aggregation

Let us consider a two-period version of the temporal-aggregation model discussed by Tiao and Wei (1976). Assume for simplicity that the true relation linking \( y_t \) and \( x_t \) is the FDL

\[
y_t = p(L)x_t + e_t, \tag{31}
\]

where \( e_t \) is orthogonal to \( y_{t-h}, x_{t-k}, h > 0, \) all \( k \) and \( p(L) = p_0 + p_1 L + \cdots + p_{2r} L^r \). However, data are available only for \( Y_T = y_{2t} + y_{2t-1} \) and \( X_T = x_{2t} + x_{2t-1} \). We are interested in the relation linking \( Y_T \) and \( X_T \).

Summing \( y_t \) and \( y_{t-1} \) gives

\[
Y_t = p(L)X_t + E_t, \tag{32}
\]

where \( Y_t = y_t + y_{t-1}, X_t = x_t + x_{t-1} \) and \( E_t = e_t + e_{t-1} \). From (32) we get

\[
Y_T = a(B)X_T + c(B)Z_T + E_T, \tag{33}
\]

where \( B = L^2 \), \( a(B) = p_0 + p_2 B + p_4 B^2 + \cdots + p_{2r} B^r \), \( c(B) = p_1 + p_3 B + \cdots + p_{(2r-1)} B^{r-1} \), \( E_T = E_{(2t)} \) and \( Z_T = X_{T-1} \). Note that in equation (33) \( E_T \) is orthogonal to \( Y_{T-2h}, X_{T-2k}, Z_{T-2k}, h > 0, \) all \( k \). Equation (33) transforms our temporal aggregation problem into an omitted-variable problem. Indeed, the misspecified relation is obtained from the true relation (33) by omitting \( Z_T \).

If \( p(L) = p_0 \), then \( c(L) = 0 \) and the aggregate equation is \( Y_T = p_0 X_T + E_T \). Therefore, if the true relation is static the aggregate relation is static. Conversely, if \( p(L) \) is not a constant and the odd coefficients of \( p(L) \) are not all zero, then \( c(B) \neq 0 \), so that by Proposition 1 a misspecification error arises and by Proposition 2 the aggregate relation is not invariant with respect to the model of \( Z_T \) and \( X_T \).

Let us assume that the Wold representation of \( x_t \) is

\[
x_t = [1/b(L)]u_t
\]
where $b(L) = 1 + b_1 L + \cdots + b_{(2s)} L^{2s}$.

It follows that

$$b(L)X_t = v_t + u_{t-1}$$

and

$$
\begin{pmatrix}
\nu(B) & \mu(B) \\
\mu(B) & \nu(B)B
\end{pmatrix}
\begin{pmatrix}
Z_T \\
X_T
\end{pmatrix}
= 
\begin{pmatrix}
1 & 1 \\
B & 1
\end{pmatrix}
\begin{pmatrix}
v_T \\
V_T
\end{pmatrix},
$$

where $\mu(B) = 1 + b_2 B + \cdots + b_{(2s)} B^s$, $\nu(B) = b_1 + b_3 B + \cdots + b_{2s-1} B^{s-1}$, $v_T = v_{(2t)}$, $V_T = v_{(2t-1)}$. Note that the residuals $v_T$ and $V_T$ are orthogonal to $X_{T-2h}$ and $Z_{T-2h}$ for $h > 0$.

Representation (5)-(6) for $Z_T$ and $X_T$ is

$$
\begin{pmatrix}
2 - b_1 \mu - \frac{1 - b_1 + B \nu}{q} \\
(1 - (1 - b_1) B) \nu - b_1 \mu
\end{pmatrix}
\begin{pmatrix}
Z_T \\
X_T
\end{pmatrix}
= \begin{pmatrix}
1 - B \\
(1 - B)
\end{pmatrix}
\begin{pmatrix}
U_{ZT} \\
U_{XT}
\end{pmatrix},
$$

(34)

where $q = 1 + (1 - b_1)^2$ and $U_{ZT} = -V_T/q - (1 - b_1)v_T/q$.

As Tiao and Wei (1976) point out, the causation structure of model (31) is destroyed upon temporal aggregation. From (34) and Proposition 4 it follows that if $c(L)$ does not vanish within the unit circle then $Y_T$ does not Granger cause $X_T$ if and only if $[1 - (1 - b_1) B] \nu(B) = b_1 \mu(B)$. This condition imposes restrictions devoid of economic meaning on the parameters $b_1, \ldots, b_{2s}$. The only interesting case is $b_h = 0$ for $h > 0$, i.e. $x_t$ is white noise. In this case $Y_T$ does not Granger cause $X_T$.

The dynamic structure of equation (31) is completely modified. The aggregate equation is not in general a FDL. Indeed, by Proposition 8 the relation linking $Y_T$ and $X_T$ is an unrestricted ARMAX unless

$$
\frac{2 - b_1}{q} \mu(B) - \frac{1 - b_1 + B}{q} \nu(B) = \frac{(1 - B)c(B)}{p_1}.
$$

(35)

Note that a necessary condition for (35) is $\mu(1) = \nu(1)$, i.e. $1 + b_2 + \cdots + b_{(2s)} = b_1 + b_3 + \cdots + b_{(2s-1)}$. If $x_t$ is white noise the latter condition is not satisfied.

---

2 The case of a general ARMA can be treated in a similar way.
CONCLUDING REMARKS

When a relevant explanatory process is omitted, the resulting misspecified equation does not preserve in general the dynamic shape and the exogeneity properties of the original relation. The parameters of the misspecified relation depend on the parameters of the model linking the omitted variable $Z_t$ and the explanatory variable $X_t$, so that the misspecified relation is not invariant with respect to policy interventions affecting the latter model. Unidirectional causation is lost, unless $Z_t$ does not Granger cause $X_t$. If the dependent variable $Y_t$ does not depend on $X_t$ in the “true” relation, it depends on $X_t$ in the misspecified relation, unless $Z_t$ and $X_t$ are orthogonal at all leads and lags. Lags of the dependent variable occur in general in the misspecified relation even though such lags do not occur in the true relation. If the true relation is static, the misspecified equation is dynamic, unless the model of $Z_t$ conditional on $X_t$ is static.

These results apply to misspecification arising from measurement errors, non-linearities, unobserved components, aggregation over agents, systematic sampling and temporal aggregation. Such kinds of misspecification destroy exogeneity and produce relations with a complex dynamic shape. Since estimated macroeconomic relations are likely to be affected by measurement errors, non-linearity, imperfect aggregation over agents and temporal aggregation, their dynamic properties are unlikely to reflect the underlying economic behaviour. Moreover, misspecification must be regarded as a major source of dynamics for macroeconomic relations.

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