Monetary policy and the term structure of interest rates: 
A generalisation of McCallum (1994) model

di

Luisa Malaguti*
Costanza Torricelli**

Maggio 1996

Università degli Studi di Modena
** Dipartimento di Economia Politica
Viale Berengario, 51
41100 Modena (Italia)
e-mail: torricelli@unimo.it

* Dipartimento di Matematica Pura ed Applicata
e-mail: malaguti@dipmat.unimo.it
Monetary policy and the term structure of interest rates:
A generalisation of McCallum(1994) model

Luisa Malaguti(*) , Costanza Torricelli(*)

Dipartimento di Matematica Pura ed Applicata, Dipartimento di Economia Politica
Università di Modena

Abstract

McCallum (1994) sets up a RE model for the interaction of monetary policy and the Expectation Theory (ET) of the term structure which rationalises some empirical failures of the latter and demonstrates the inappropriateness of usual regression tests for the information in the term structure. In the present paper, we generalise McCallum (1994) two-period model by introducing a different, finance-theoretic characterisation of the term premium. Our results still account for those empirical findings which are at odds with the ET of the term structure and yet support validity of usual regressions performed to assess the information content of the term structure.

(*) I acknowledge financial support from GNAFA-CNR and MURST 40% “Analisi reale”.
(*) The paper was completed while I was visiting at the Institut für Entscheidungstheorie und Unternehmensforschung of the University of Karlsruhe. I wish to thank Hermann Göppel and all the members of the Institut for providing me with research support. I also acknowledge financial support from MURST 40% 94-95 and CRUI (Progetto Vigoni). Usual caveats apply.
1. Introduction

The term structure of interest rates has been recently object of investigation by means of sophisticated stochastic models, which are mainly aimed at pricing interest sensitive derivative securities\(^1\). Yet, it should not be forgotten that the term structure plays a role not only in finance, but also in economic theory.

In fact, the term structure has interesting implication for economic policy since it provides the relationship between short term interest rates, which are normally considered as policy variables, and long term interest rates, which typically influence macroeconomic functions.

The term structure is therefore interesting in itself for its information content, which has been object of accurate empirical investigation in the last decade (among others: Campbell and Shiller(1984, 1991), Fama(1984, 1990), Mishkin(1990), Rudebusch(1995) and for a survey Torricelli(1995))

A related and interesting issue is represented by the interaction between monetary policy and the term structure of interest rates. The basic idea of the present paper is to investigate what are the effects of monetary policy on the term structure of interest rates (TS), with particular attention to the predictive content of the TS.

The issue has already been analysed in the eighties and has been recently reproposed by McCallum(1994). The author takes as starting point the empirical failures of the Expectation Theory (ET) of the TS and proposes an explanation based on the existence of time varying term premia and an "interest rate smoothing" policy rule according to which the central bank changes the target (the short rate) in response to movements in the term spread. On the basis of the results obtained, McCallum concludes that most regression tests conducted on the TS in order to detect its information content are not appropriate. As it stands, the model should be "regarded as more of a parable than a fully-worked-out quantitative model" and the author suggests to study the effects of the introduction of different monetary policy regimes.

In the present paper, we mean to take a line of investigation of the empirical failure of the ET centred on the role played by time-varying term premia. That the failure of the ET is due to the existence of time-varying term premia has been since long recognised. Yet, only recently term premia have been determined endogenously within general equilibrium stochastic models and some light has been shed on their time-varying nature.

The model we propose here is essentially a variation of McCallum two-period model, based on a different characterisation of the time-varying nature of the term premium which offers quite different results and has different implications as for the predictive content of the term structure.

In the following section we describe McCallum(1994) two-period model. In section 3, we propose our modification of the model and we discuss the results obtained. Details about the solutions and the procedure used are given in the Appendix. Conclusions and further lines of investigation are provided in section 4.

2. McCallum(1994) two-period model

In this section, we will briefly recall the main equations of McCallum's two-period model and those results relevant to the comparison with the model we propose in the next section.\(^2\) For details about the Rational Expectation (RE) solution procedure, we refer the interested reader to the original papers, i.e. McCallum(1983, 1994).

According to the ET, the equation for the TS in a two-period model is:

---

\(^1\) For surveys on stochastic models of the term structure of interest rates, see Vetzal(1994) and Boero and Torricelli(1996).

\(^2\) Given the absence of nominal variables in the model, in this paper interest rates should be thought of as real interest rates.
\[ R_t = \frac{1}{2}(r_t + E_{r_{t+1}}) + \xi_t \]  

(1)

Where:
- \( r_t \) is the one-period return on the one-period zero coupon bond,
- \( R_t \) is the one-period return on the two-period zero coupon bond,
- \( \xi_t \) is the term premium.

Equation (1) can be also written in the following way:

\[ \frac{1}{2}(E_{r_{t+1}} - r_t) = (R_t - r_t) - \xi_t \]  

(1a)

Which can in turn be transformed in the corresponding statistical equation:

\[ \frac{1}{2}(r_{t+1} - r_t) = (R_t - r_t) - \xi_t + 1/2\varepsilon_{t+1} \]  

(1b)

With \( \varepsilon_{t+1} \) being the expectational error which is orthogonal to \( r_t \) and \( R_t \) and defined as:

\[ \varepsilon_{t+1} = r_{t+1} - E_r r_{t+1}. \]

Tests on the validity of the ET have been based on the estimation of the parameter of the following equation:

\[ \frac{1}{2}(r_t - r_{t-1}) = a + b(R_{t-1} - r_{t-1}) + \text{disturbance} \]  

(1c)

According to the ET, \( b \) should be equal to 1, \( \alpha \) should be equal to the constant term premium. The issue has already been long investigated in the empirical literature. In particular, the characterisation of the term premium as being constant, has been considered responsible for the empirical failure of the very same theory. In order to take this critique into account, McCallum assumes the term premium to be autoregressive of order 1, i.e.:

\[ \xi_t = \rho \xi_{t-1} + u_t \]  

(2)

where \( u_t \) is a white noise and \( |\rho| < 1 \).

The monetary policy rule is supposed to be aimed at interest rates smoothing, specifically it is assumed that:

\[ r_t = \sigma r_{t-1} + \lambda (R_t - r_t) + \zeta_t \]  

(3)

with \( \sigma > 0 \), which is presumed to be close to 1, and \( \lambda > 0 \), to be smaller than or equal to 2\(^4\) and \( \zeta_t \) is a white noise.

Equations (1), (2) and (3) represent only a portion of a macroeconomic system. Yet, if the two white noise terms are assumed to be independent of the remaining relations, the system is recursive and can be solved for \( r_t \).

The fundamental bubble-free RE solution is attained by means of the Minimal State Variable (MSV) criterion proposed in McCallum(1983).

---

3 The point has long discussed: among others, see Campbell and Shiller(1984, 1991).
4 McCallum(1994) in footnote (9) writes that the condition is imposed because "it is plausible and also because a theoretical issue concerning the root of (10) that gives the bubble-free solution, arises when \( \lambda > 2 \)."
Combining (1)-(3), yields the following equation:

$$(1 + \lambda)r_t = \sigma r_{t-1} + \lambda \left[ \frac{1}{2} (r_t + \epsilon_t + \xi_t) \right] + \zeta_t$$

(4)

The MSV criterion suggests to find a solution of the following form:

$$r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 \xi_{t-1} + \phi_3 \zeta_t$$

(5)

From (5) it is easy to work out the expression for $E_{sr_{t+1}}$, which substituted, together with (5), into (4) leads to a non-linear system of four equations in the undetermined coefficients $\phi_0, \phi_1, \phi_2, \phi_3$. Under reasonable assumptions, such a system has a unique solution. For the case of $\sigma=1$, the values of the above coefficients simplify to:

$$\phi_0 = 0, \phi_1 = 1, \phi_2 = \frac{\lambda}{1 - \rho \lambda / 2}, \phi_3 = 1$$

So that the solution for the one-period rate becomes:

$$r_t = r_{t-1} + \frac{\lambda}{(1 - \rho \lambda / 2)} \xi_{t-1} + \zeta_t$$

(6)

and, accordingly, the spread and the first difference in interest rates are respectively given by:

$$R_t - r_t = 1/2 (E_t r_{t+1} - r_t) + \xi_t = (1 - \rho \lambda)^{-1} \xi_t$$

(7)

$$r_t - r_{t-1} = \frac{\lambda \rho}{1 - \lambda \rho / 2} \xi_{t-1} + \frac{\lambda}{1 - \lambda \rho / 2} u_t + \zeta_t$$

(8)

Combining (7) and (8) gives:

$$\frac{1}{2} (r_t - r_{t-1}) = \frac{\lambda \rho}{2} (R_{t-1} - r_{t-1}) + \frac{\lambda / 2}{1 - \lambda \rho / 2} u_t + 1/2 \zeta_t$$

(9)

where the two disturbances are uncorrelated with the spread.

On the basis of this result, McCallum comments upon the appropriateness of those tests on the information content of the ET essentially based on the estimation of equation (1c). Were $\rho$ or $\lambda$ equal to zero (i.e. if the term premium were a white noise or if the monetary policy rule were simply autoregressive), the slope coefficient in eq. (9) would also be zero. Such a result "demonstrates, I would suggest, not only that the usual regression test is inappropriate but also that it is misleading to think of the expectation theory in terms of the "predictive content" of the spread for future changes of the short rate. Such a predictive content is not a necessary implication of the theory". (McCallum(1994), page 7).

The author proposes, as a further line of investigation of the issue, the need of considering different monetary policy rules.

In our opinion, McCallum's result rests not only upon the specific assumptions made on the monetary policy rule, but also on the time-varying behaviour assumed for the term premium. In the next section, we propose a model with a different characterisation of the term premium and we derive the RE solutions and discuss some implications under such an assumption.

---

5 Cfr. McCallum(1994) page 7 and McCallum(1983): the basic idea is to choose only that solution which is valid for all admissible values of the structural parameters.
3. An alternative two-period model

The model we propose in this section differs from McCallum’s for the characterisation of the term premium. McCallum recognises the need of a time-varying term premium and simply models it as an AR(1) process.

The time-varying nature of term premia in relation to the ET empirical failures has been long debated in the literature. The point is clearly made by Mankiw and Summers(1984): "Without an explicit theory of why there is such a premium and why it varies, it has no function but tautologically to rescue the theory. [...] These results suggest the importance of developing models capable of explaining fluctuating liquidity premiums. [...] Without a satisfactory theory of liquidity premiums, predicting the effect of policies on the shape of the yield curve is almost impossible."

Hence the question is now the following: is it a theory of the term premium available in the literature? In our opinion, a positive answer to this question is to be found within general equilibrium stochastic models of the term structure of the type introduced by Cox, Ingersoll and Ross (1985), CIR, and further developed by Longstaff and Schwartz (1992), LS. Each paper offers a precise functional form for the risk premium, which in both cases turns out to be a rather cumbersome function involving the model parameters.

Yet, to the aim of the present paper, a useful implication of the two mentioned stochastic models of the term structure (CIR and LS) is that the time-varying nature of the term premium is due its dependence on the short term interest rate.6

In what follows, we will model the dependence of the term premium on the short rate in such a way as to preserve linearity of McCallum’s model and to allow, at the same time, a sensible economic interpretation. To this end, we have decided to model the term premium as follows:

\[
\xi_t = \alpha (E_{t+1} r_{t+1} - r_{t-1}) + \beta_t \quad \text{with} \quad \beta_t = \rho \beta_{t-1} + u_t
\]

where \( \alpha \geq 0, \rho < 1 \) and \( u_t \) is a white noise.

The term into brackets can be thought of as:

\[
\left( (E_{t} r_{t+1} - r_t) + (r_t - r_{t-1}) \right) / 2
\]

and hence represents the simple arithmetic mean of the changes in the short rate over the two periods under investigation in the present model. Therefore, it can be interpreted as the agents measures of the variation in the short rates based on what happened in the past (between \( t-1 \) and \( t \)) and what is expected to happen (between \( t \) and \( t+1 \)).

Eq. (10) essentially says that the term premium has two components: one is autoregressive (the second term on the r.h.s.) and has the same characteristics of McCallum’s term premium, the other captures the dependence of the premium on the short term rate. When the short interest rate is expected to remain constant, the term premium is purely autoregressive and the model would collapse in McCallum’s (which would also be the case if \( \alpha = 0 \)). The term premium is bigger (smaller) than its pure autoregressive component if the short rate is expected to increase (decrease). Intuitively, such a characterisation of the term premium is consistent with its very existence: in a risk-averse setting, long rates are requested to be bigger than the average between the current and the expected future spot rate, when short rates are expected to rise and viceversa.

Since the other equations are the same as in McCallum’s, our model is fully described by equations (1), (10) and (3), where the first two together provide a behavioural equation for the TS and the latter describes the monetary policy rule.7

---

6 LS conclude that, for a fixed maturity, the term premium is a linear function of the riskless interest rate and its volatility. Yet, the inclusion of a volatility measure would transform the basic model into a non-linear one.

7
Combining (1), (10) and (3) yields to the following equations:

\[(1 + \lambda / 2)r_t = (\sigma - \alpha \lambda)r_{t-1} + \lambda (1 / 2 + \alpha) E_{t+1} + \lambda \beta_t + \zeta_t.\] (11)

Hence, according to the MSV criterion, we also look for a solution of the following form\(^8\):

\[r_t = \phi_1 r_{t-1} + \phi_2 \beta_t + \phi_3 \zeta_t.\]

Using the procedure proposed by McCallum(1983) and under reasonable assumptions on the parameters values\(^9\), we obtain the following RE solution for the short rate:

\[r_t = \frac{1}{1 + 2\alpha} r_{t-1} + \frac{\lambda}{1 - \alpha \lambda - \lambda^2 / 2} \beta_t + \zeta_t, \text{ with } 0 \leq \lambda \leq 2\] (12)

Accordingly, the spread and the first difference in the short rate are given by:

\[R_t - r_t = -\alpha r_{t-1} + \frac{1}{1 - \alpha \lambda - \lambda^2 / 2} \beta_t,\] (13)

\[\frac{1}{2} (R_t - r_{t-1}) = \frac{1}{(1 + 2\alpha)^2} (R_{t-1} - r_{t-1}) + \frac{\lambda \rho + 4\alpha^2 \lambda \rho + 4\alpha \lambda \rho - 2\alpha \lambda - 4\alpha^2 \lambda - 2}{(1 + 2\alpha)^2 (2 - 2\alpha \lambda \rho - \lambda \rho)} \beta_{t-1}\]

\[-\frac{\alpha}{1 + 2\alpha} \zeta_{t-1} + \frac{\lambda}{2 - 2\alpha \lambda \rho - \lambda \rho} u_t + \frac{1}{2} \zeta_t.\] (14)

A few comments are in order. First of all, eq. (12) and (13) collapse into (6) and (7) respectively for \(\alpha = 0\), which means that our model contains McCallum’s as a special case.\(^10\)

As for the spread between the long and the short rate, by comparison of eq. (7) and (13), it is clear that, in our model, it is not only a function of the autoregressive component of the term premium, but also of the level of the short term interest rate. In particular, the RE spread at time \(t\) depends negatively on the level of the short interest rate a period before: the result stems from the assumption we have made about the term premium and can intuitively be explained by the fact that the latter is assumed to be negatively dependent on \(r_{t-1}\).\(^11\)

But the most interesting results are to be deduced from eq. (14). In order to illustrate them, we will first make some observations on the characteristics of eq. (14) and then we comment upon the implications for the analysis of the information content of the TS.

---

\(^7\) The assumption taken on the model parameters are, beyond those on \(\alpha\), the same as in McCallum’s: possible further restrictions, are eventually discussed in the Appendix and may be useful in the interpretation of the results of the present section.

\(^8\) We omit the constant \(\phi_0\), since it turns out to be, both in McCallum(1994) and in our model, always equal to zero.

\(^9\) Details on the solution procedure are given in the Appendix. For simplicity, we impose the relationship expressed by (A4) among the model parameters. Yet, by continuity, small deviations from such a relationship do not imply qualitative changes in the solutions.

Note also that the restriction imposed on \(\lambda\) is the same as in McCallum’s model and has the same rationale, i.e it serves to single out only one value for \(\phi_0\). It should also be stressed that such a restriction on \(\lambda\) is also economically plausible and desirable since it imposes a limit on the dependence of the short rate on the spread in the monetary policy rule.

\(^10\) The same cannot be said for eq. (14) since its derivation implies that \(\alpha \neq 0\).

\(^11\) According to eq. (10), in our model the term premium is actually a function of \([E(t_{t+1}) - r_{t-1}]\). Easy calculations show that the RE value of such a difference is itself a negative function of \(r_{t-1}\).
Looking at (14) it is clear that: i) the coefficient of the spread is a function of $\alpha$ and does not to include anymore the other parameters (such as $\lambda$ and $\rho$), ii) the autoregressive part of the term premium ($\beta_{t-1}$) comes into the scene as a separate regressor, iii) the same is true for the white noise term in the monetary policy rule ($\xi_t$).

The first two characteristics of eq. (14) stem from the new characterisation of the term premium (and hence of the spread). That $\alpha$ plays a role in equation (14) is not surprising, given the assumption made on the term premium. As for $\lambda$ and $\rho$, they enter only through the autoregressive component of the TP (second term on the r.h.s. in eq. (13)), which is contained in the spread.

In McCallum (see eq. (9)) the autoregressive component of the TP does not explicitly appear as a regressor because there is a substantial coincidence between the TP and the spread and the latter is normally taken as regressor since it lends itself to a more interesting interpretation. Instead, eq. (9) is derived by substituting eq. (7) into (8), which represents the change in the short rate as a function of the autoregressive part of the TP.

In order to better understand the difference between the two models, it is convenient to subtract $r_{t-1}$ from each side of eq. (12), which gives:

$$ (r_t - r_{t-1}) = \frac{-2\alpha}{(1+2\alpha)} \beta_{t-1} + \frac{\lambda \rho}{1 - \alpha \lambda \rho - \frac{\lambda}{2} \rho} \beta_{t-1} + \frac{\lambda}{1 - \alpha \lambda \rho - \frac{\lambda}{2} \rho} u_t + \xi_t, \quad (15) $$

Eq. (15) is the equivalent of McCallum eq. (8) and, in fact, for $\alpha = 0$ collapses in eq. (8).

By comparison of eq. (8) and eq. (15), it is clear that the role played by the autoregressive TP is the same in the two models. Yet, our characterisation of the TP introduces a further determinant of the one-period variation in the short rate, the first term on the r.h.s. of eq. (14) or (15), which, under the monetary policy rule taken in this paper, is still active even if the policy parameter $\lambda$ were set to zero.

We think that the most interesting features of the results obtained lie in their implications for the investigation of the information content of the TS within the ET framework.

First of all, in contrast with McCallum (1994) conclusions\(^{12}\), neither $\rho = 0$ nor $\lambda = 0$ implies a null coefficient for the spread in eq. (14). Thus the appropriateness of regression tests of the type implied by (14) is maintained.

Hence, our results demonstrate that McCallum’s statement about the inappropriateness of using usual regression tests to detect the information content of the TS stems from the very specific assumptions he takes on the time-varying term premium and in particular from its purely autoregressive nature.

Secondly, eq. (14) provides a theoretical justification of a widespread empirical result: the slope coefficients emerging from estimations of eq. (1c) tend to be well below 1. Such a result has been often claimed to be inconsistent with either the ET or one of the maintained hypotheses (RE, constant term premium). The introduction of a time-varying term premium of an autoregressive type as in McCallum’s model does not solve the puzzle. Our alternative model provides a justification of the most common empirical finding: the coefficient of the term premium is smaller than 1, simply because the term premium is not purely autoregressive, but depends also on the past and expected behaviour of the short term interest rate (i.e. $\alpha = 0$ and specifically $\alpha > 0$).

Finally eq. (14) suggests that, in contrast with McCallum’s, the spread between the long and the short interest rate is not the only determinant of the change in the short rate. In order to better understand this point, let us take McCallum’s model and assume that $\rho$ is positive, given the assumption on $\lambda$, the coefficient of the spread in eq. (9) is positive, which means that a positive spread implies an increase in the short rate and vice versa. In eq. (14) such an effect, which is

\(^{12}\) See McCallum(1994) page 7 or the citation reported at the end of section 2.
consistent with the ET, is still present, but it might be dumped or reinforced by the second and the third term on the r.h.s. Specifically, while the latter should be modest in dimension, the former deserves some more comments.

Let us suppose that at some time, the spread \((R_{t+1} - r_{t+1})\) and \(\beta_{t+1}\) are positive. As in McCallum’s, the positive spread predicts an increase in the short rate in the next period. As for \(\beta_{t+1}\), since its coefficient can be, under reasonable assumption on the parameter values, negative (see Appendix for a proof), it follows that a positive \(\beta_{t+1}\) predicts a decrease in the short rate. The total effect is therefore indeterminate and is eventually an empirical question.

Intuitively, we can think of eq. (15) as obtaining a sort of separation of the effects that the ET assumption and the interest rates smoothing monetary policy rule respectively have on the change in the short rate. The first term on the r.h.s. captures the implication of that part of the spread that works out its effects through rates (and hence independently of \(\lambda\)); the second captures the effects of the autoregressive part of the TP via the interest rate smoothing part of the policy rule (and hence disappears when \(\lambda\) or \(\delta\) are zero). The total effects depends on the parameters values.

Concluding on this last point, we think that having the spread as the single regressors in most empirical tests of the information in the TS, implies that some information contained in the TS is missed and this may in turn explain poor results in terms of \(R^2\) (e.g. Fama (1984,1990)).

4. Conclusions

Starting point of the present paper is McCallum (1994) two-period model for the interaction between the ET of the term structure of interest rates and a policy rule involving interest rates smoothing. The author shows that anomalous empirical findings, about the magnitude of the slope coefficients in regressions of short rates over the long-short spread, can be explained by consideration of an exogenous autoregressive term premium. Yet, the author suggests such usual regressions are inappropriate to detect the predictive content of the TS spread since the spread coefficients becomes zero whenever the monetary policy rule parameter \((\lambda)\) or the autoregressive term premium parameter \((\rho)\) are set to zero.

In this paper we show that McCallum’s negative conclusion about information tests of the TS derives from having assumed the term premium to be purely autoregressive. By characterising the term premium as depending not only on an autoregressive component but also on the past and expected future short rate, we obtain a regression equation where the coefficient of the TS spread does not go to zero as \(\lambda\) or \(\rho\) are equal to zero.

Moreover, our model is able to rationalise other two widespread empirical findings. First of all, due to the new assumption taken on the term premium (i.e. \(\alpha = 0\)), the spread coefficient is smaller than one: a result which is consistent with most empirical investigations (e.g. Fama(1984,1990)). Secondly, the presence of regressors other than the spread in our main regression equation provides a rationale for the low \(R^2\) characterising most empirical studies: using only the spread as a regressor some information contained in the term structure is missed.

For \(\alpha = 0\), we get McCallum model and results as a special case and hence our model can be considered as a generalisation of McCallum model.

The present paper has to be seen as preliminary to a deeper investigation of the interaction between monetary policy and the term structure of interest rates and lends itself to be extended in many directions.

The first natural extension is to the n-period case. McCallum(1994) actually shows the the change in the short rates displays qualitatively the same characteristics as in the two-period model. Yet, the n-period model allows for the investigation of the information content of the spread as for the long (n-period) rate.
Secondly, given the sensitivity of the result obtained to the monetary policy rule assumed, it would be interesting to analyse the effect of alternative, and possibly more realistic, policy rules (see Rudebusch (1995) page 247).

Finally, the absence of nominal variable in the models analysed in the present paper, naturally raises the opportunity of bringing the price level into the model. The latter could also, in some time- and country-specific set up, play a role in the monetary policy rule.

Appendix

Here the purpose is to describe the procedure used to obtain the difference in the short rate given by (14) and to discuss the assumptions on the parameters needed to this aim.

In order to get a unique *bubble-free* solution for the short rate dynamic, we argue that the agents employ a minimal set of state variables for their forecasts; such method, suggested by McCallum (1983), is known as the minimal-state-variable (MSV) criterion.

As one can see in the fundamental relation of our model, i.e. equation (11), the short rate \( r_t \) depends only on the state variables \( r_{t-1}, \beta_t \) and \( \zeta_t \); consequently, the equation for \( r_t \) will be a linear function of such variables, that is:

\[
 r_t = \phi_1 r_{t-1} + \phi_2 \beta_t + \phi_3 \zeta_t, \tag{A-1}
\]

with \( \phi_1, \phi_2, \phi_3 \) undetermined coefficients.

The expectational variable is then given by:

\[
 E_t r_{t+1} = \phi_1^2 r_{t-1} + (\phi_1 + \rho) \phi_2 \beta_t + \phi_1 \phi_3 \zeta_t,
\]

and substituting into (11) one has:

\[
 1 + \frac{\lambda}{2} \left( \phi_1 r_{t-1} + \phi_2 \beta_t + \phi_3 \zeta_t \right) = (\sigma - \alpha \lambda) r_{t-1} + \lambda \left( \frac{1}{2} + \alpha \right) \phi_1^2 r_{t-1} + \lambda \left( \frac{1}{2} + \alpha \right) \phi_1 + \rho \phi_2 \beta_t + \lambda \left( \frac{1}{2} + \alpha \right) \phi_1 \phi_3 \zeta_t + \lambda \beta_t + \zeta_t.
\]

For (A-1) to be a solution for the short rate the following identities in \( r_{t-1}, \beta_t \) and \( \zeta_t \) must be true:

\[
\begin{align*}
 1 + \frac{\lambda}{2} \phi_1 &= \sigma - \alpha \lambda + \lambda \left( \frac{1}{2} + \alpha \right) \phi_1^2, \\
 1 + \frac{\lambda}{2} \phi_2 &= \lambda \left( \frac{1}{2} + \alpha \right) \phi_1 + \rho \phi_2 + \lambda, \\
 1 + \frac{\lambda}{2} \phi_3 &= \lambda \left( \frac{1}{2} + \alpha \right) \phi_1 \phi_3 + 1
\end{align*} \tag{A-2}
\]

In order to solve the system it is convenient to distinguish the two cases \( \lambda = 0 \) and \( \lambda \neq 0 \). Let us begin with the latter one \( \lambda \neq 0 \); the first equation in (A-2) yields:
\[
\phi_1 = \frac{1 + \frac{\lambda}{2} \pm \sqrt{(1 + \frac{\lambda}{2})^2 - 4 \lambda \left(1 + \frac{\lambda}{2}\right)(\sigma - \alpha \lambda)}}{\lambda(1 + 2\alpha)}.
\] (A-3)

In order to choose between the two possibilities for \(\phi_1\), assume \(\sigma = 0\) and \(\alpha = 0\); in this particular case \(r_t\) does not depend on \(r_{t-1}\), as one can see in (11) so \(r_{t-1}\) has not to be included in the minimal set of state variables and the value for \(\phi_1\) is zero; but the zero root is obtained only with the negative square root in (A-3). As in McCallum (1983) we require that the solution formula for \(\phi_1\) must be valid for all admissible values of the parameters, hence we take the negative determination in (A-3) for all \(\alpha, \lambda\) and \(\sigma\).

It is convenient to introduce the following relation on the parameters:

\[
\sigma = \alpha \lambda + \frac{1}{1 + 2\alpha};
\] (A-4)

in this case, in fact, the coefficient \(\phi_1\) has a very simple expression so it is easier to analyse the implications of such solution.

At the end of this part we will discuss the special case \(\sigma = 1\), which is suggested by the interest rate smoothing behavior and is also discussed in McCallum (1994).

When (A-4) holds, from (A-3) and the condition \(0 < \lambda \leq 2\) we get \(\phi_1 = \frac{1}{1 + 2\alpha}\) and assuming:

\[
2 - \lambda \rho (1 + 2\alpha) \neq 0
\] (A-5)

from the second and third equation of (A-2) we obtain:

\[
\phi_2 = \frac{\lambda}{1 - \alpha \lambda \rho - \lambda \rho / 2}, \quad \phi_3 = 1;
\]

according to (A-1), the solution for the short rate \(r_t\) is then given by (12). Consequently the expected value of \(r_{t-1}\) becomes

\[
E_t r_{t-1} = \frac{1}{1 + 2\alpha} r_t + \frac{\lambda \rho}{1 - \alpha \lambda \rho - \lambda \rho / 2} \beta_t;
\]

if we substitute such relation in (1) and take into account (10), we find that the spread obeys equation (13), so in particular it holds:

\[
R_{t-1} - r_{t-1} = -\alpha r_{t-2} + \frac{1}{1 - \alpha \lambda \rho - \lambda \rho / 2} \beta_{t-1},
\] (A-6)

Of course, the solution for the short rate (12) may also be referred to the time \(t-1\); if we get \(r_{t-2}\) from such relation we obtain

\[
r_{t-2} = (1 + 2\alpha) r_{t-1} - \frac{\lambda(1 + 2\alpha)}{1 - \alpha \lambda \rho - \lambda \rho / 2} \beta_{t-1} - (1 + 2\alpha) \zeta_{t-1}
\]

and combining with (A-6) it yields
\[ R_{t-1} - r_{t-1} = -\alpha (1 + 2\alpha) r_{t-1} + \frac{\alpha \lambda + 2\alpha^2 \lambda + \lambda}{1 - \alpha \lambda \rho - \lambda \rho / 2} \beta_{t-1} + \alpha (1 + 2\alpha) \zeta_{t-1} \]

Whenever \( \alpha \neq 0 \), we are able to express \( r_{t-1} \) in previous equation as a function of the spread at time \( t-1 \) and of \( \beta_{t-1} \) and \( \zeta_{t-1} \); more precisely we have:

\[
r_{t-1} = -\frac{1}{\alpha (1 + 2\alpha)} (R_{t-1} - r_{t-1}) + \frac{\alpha \lambda + 2\alpha^2 \lambda + \lambda}{\alpha (1 + 2\alpha)(1 - \alpha \lambda \rho - \lambda \rho / 2)} \beta_{t-1} + \zeta_{t-1}
\]

and substituting in (12) we get the final relation (14) concerning the first difference in the short rate.

On the other hand, when \( \lambda = 0 \), from (A-2) we have \( \phi_1 = \sigma, \phi_2 = 0 \) and \( \phi_3 = 1 \), so if we assume for \( \sigma \) the special value (A-4) which reduces to \( \frac{1}{1 + 2\alpha} \), (12) continue to be true and consequently also (14) holds.

It is worth now to discuss the sign of the coefficient of \( \beta_{t-1} \) in (14).

Easy calculations lead to the following result:

\[
\frac{\lambda \rho + 4\alpha^2 \lambda \rho + 4\alpha \lambda \rho - 2\alpha \lambda - 4\alpha^2 \lambda - 2}{(1 + 2\alpha)^2 (2 - 2\alpha \lambda \rho - \lambda \rho)} \geq 0 \quad \text{if and only if}
\]

\[
\lambda \neq 0 \quad \alpha > \frac{2 - \lambda}{2\lambda} \quad \text{and} \quad \frac{2 + 2\alpha \lambda (1 + 2\alpha)}{\lambda (1 + 2\alpha)^2} \leq \rho < \frac{2}{\lambda (1 + 2\alpha)}.
\]

In fact, when \( \lambda = 0 \) such coefficient reduces to \( -\frac{1}{(1 + 2\alpha)^2} \), so it is negative.

In case \( \lambda \neq 0 \) and \( 0 < \alpha \leq \frac{2 - \lambda}{2\lambda} \), the coefficient would be positive outside the admissible range of values for \( \rho \).

Finally we briefly comment upon relation (A-5) which introduces, when \( \rho \) is positive, a further condition on the parameters in the model. More precisely we want to show that, whenever \( \sigma \) is sufficiently near 1, such condition always holds. Indeed (A-5) does not depends explicitly on \( \sigma \); but, in view of (A-4), it depends on \( \sigma \) through \( \alpha \). In (A-4) it is possible to express \( \alpha \) as a function of \( \sigma \); in fact, we have:

\[
\frac{1}{\sigma} = \frac{1}{2} \left( \frac{1}{\sigma} - 1 \right) \quad \text{for} \ \lambda = 0,
\]

\[
\sigma - 1 + \sqrt{\sigma^2 + 2\sigma - 3} \quad \text{for} \ \lambda = 2 \quad \text{and}
\]
\[
\frac{2\sigma - \lambda \pm \sqrt{2\sigma - \lambda})^2 - 8\lambda(1 - \sigma)}{4\lambda} \quad \text{for } 0 < \lambda < 2
\]

with \( \sigma \) varying in appropriate domains always containing the value 1. Consequently the function appearing in (A-5) continuously depends on \( \sigma \) for all possible \( \lambda \).

If we assume \( \sigma = 1 \), we get \( \alpha = 0 \) or \( \alpha = \frac{2 - \lambda}{2\lambda} \); for both \( \alpha \) the function in (A-5) is positive, hence condition (A-5) holds at least for all values of \( \sigma \) sufficiently near 1.
REFERENCES


Torricelli C., 1995, The information in the term structure of interest rates: can stochastic models help in resolving the puzzle?, Materiale di Discussione, Dipartimento di Economia Politica, Università di Modena, n. 113, luglio.

5. Paolo Bossi e Paolo Silvestri [1986] "La distribuzione per aree disciplinari dei fondi destinati ai Dipartimenti, Istituti e Centri dell’Università di Modena: una proposta di riforma", pp. 25
7. Paolo Silvestri [1986] "Le tasse scolastiche e universitarie nella Logge Finanziaria 1886", pp. 41
24. Fernando Vianello [1987] "Effective Demand and the Rate of Profit. Some Thoughts on Marx, Kalecki and Sraffia", pp. 41
27. Giovanna Procesi [1985] "The State and Social Control in Italy During the First World War", pp. 18
43. Giovanna Procesi [1989] "State surplus and weaker solidarity in Italy (1915-1918): the moral and political content of social unrest", pp. 41
44. Carlo Alberto Magni [1989] "Reputazione e credibilità di una riforma in un gioco bargaining", pp. 56
47. Paolo Bossi, Roberto Golinelli e Anna Stagni [1989] "Le origini del debito pubblico e il costo della stabilizzazione", pp. 26
52. Paolo Silvestri [1989] "Il bilancio dello stato", pp. 34
55. Paolo Silvestri [1990] "Sull’autonomia finanziaria dell’università", pp. 11
118. Mario Fossi e Marco Lippi [1995] “Permanent income, heterogeneity and the error correction mechanism.” pp. 21
128. Carlo Alberto Magni [1996] “Repetatabile and a variation real options a dynamic programming approach” pp. 23
134. Margherita Russo, Peter Börkey, Emilio Cubel, François Lévéque, Francisco Mas [1996] “Local sustainability and competitiveness: the case of the ceramic tile industry” pp. 66