Recursive VAR Orderings and Identification of Permanent and Transitory Shocks

by

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**Abstract.** In this paper it shown that, given a bivariate VAR process which includes I(1) variables, if we assume one-way causality in the long-run, then by imposing a recursive VAR ordering it is possible to identify a permanent and a transitory shock which affect the variable ordered first. Thus, in this case, the traditional identification scheme based on contemporary restrictions (cf. Sims, 1980) and the Blanchard-Quah model (1989), with long-run restrictions, are equivalent. This result suggests that economic models which predict long-run neutrality have a structural recursive VAR representation.

*Keywords:* Structural VAR; Identification; Wold causal chain; Long-run restriction

*J.E.L. Classification:* C32; E32
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1. Introduction

In recent years Vector autoregressive (VAR) models have been widely used in empirical research. However, an important question concerns the set of identifying restrictions which need to be imposed on the reduced-form VAR representation, in order to give a structural interpretation of the disturbances affecting the economic system.

Sims (1980) proposed obtaining orthogonal disturbances via the Cholesky decomposition. Thus, given orthonormal innovations, identification is achieved by imposing contemporary restrictions on interaction among the variables.

This approach has been criticized (e.g. Cooley and LeRoy, 1985; Leamer, 1985) because it rests on the implicit assumption that the economic model is recursive. Hence, if the true model is not recursive, then the Cholesky decomposition does not provide a structural interpretation of the disturbances (cf. Bernanke, 1986).

A different approach, pioneered by Bernanke (1986), Blanchard-Watson (1986) and Sims (1986), has aimed at achieving identification by imposing structural restrictions on contemporaneous short-run effects of the shocks. This approach advocates economic theory in

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order to select credible identifying restrictions.

More recently, Blanchard and Quah (1989) proposed restrictions based on economic models which suggest long-run neutrality. Thus, given a bivariate VAR model including the first difference of GNP and the rate of unemployment, Blanchard and Quah identify two structural shocks: one, interpreted as a supply shock, has a permanent effect on the level of GNP, the other, interpreted as a demand shock, has no long-run effect.

In a recent paper, Granger and Lin (1995) discuss the notion of (Granger) causality at different frequencies. The interesting fact is that given two series, $x_{1t}$ and $x_{2t}$, $x_{2t}$ could be an important force explaining $x_{1t}$ but it may be that there is no Granger-causality in the long-run. Moreover, the above authors show that the interpretation of causal relationships between two integrated series at very low frequencies is fairly simple, depending on a few coefficients of the reduced form.

The main purpose of this paper is to show that, given a bivariate VAR process including a pair of I(1) series, there is a close link between non-causality in the long-run and existence of recursive VAR orderings which provide a structural interpretation of the disturbances affecting the economic system. In fact, if $x_{2t}$ does not cause $x_{1t}$ at frequency 0, it is possible to identify a permanent and a transitory shock, which affect $x_{1t}$, by imposing a Wold causal chain with the variable $x_{1t}$ ordered first.

Furthermore, if we assume cointegration, a shock to the variable ordered second which does not contemporaneously affect the variable first, exhibits only transitory effects on the level of both $x_{1t}$ and $x_{2t}$.

These results have an important corollary: the traditional identification scheme, based on contemporary restrictions, and the identification scheme based on long-run restrictions are equivalent, provided that the variable which is first in the causal ordering is not caused at frequency zero by the second variable.

The paper is organized as follows: section 2 introduces notation and assumptions; section 3 shows that Sims orthogonalization, given the assumption of unidirectional causality in the long-run, allows the identification of a permanent and a transitory shock for the variable which is first in the causal ordering; section 4 deals with cointegrated systems; section 5 concludes and some implications for empirical research are drawn.
2. Notation, definitions and assumptions

Let us consider the zero mean vector stochastic process $X_t = (x_{1t}, x_{2t})'$. We make the following assumption:

Assumption A1. The vector $X_t$ includes $I(1)$ series which do not exhibit a long-run equilibrium relationship. Then the change in $X_t$ has reduced-form Wold representation given by:

$$\Delta X_t = C(L)e_t$$  \hspace{1cm} (1)

where: (i) $L$ is the lag operator; (ii) $\Delta = 1 - L$ is the difference operator; (iii) $C(0) = I$; (iv) $\det C(L)$ has no zeros of modulus less than or equal to unity; (v) the spectral density matrix of $\Delta X_t$ evaluated at frequency 0, $(2\pi)^{-1}C(1)\Omega_e C(1)'$, has full rank; (vi) $e_t$ is the $(2 \times 1)$ vector of disturbances such that $E(e_t) = 0$ and $E(e_t e_t') = \Omega_e$.

Thus, both $x_{1t}$ and $x_{2t}$ exhibit a stochastic trend and have spectral density which does not vanish at zero frequency. Condition (iv) guarantees that the disturbances lie in the space spanned by current and past values of $X_t$. Hence, the vector $e_t$ is fundamental for $X_t$ (cf. Lippi-Reichlin, 1993). Moreover, (v) ensures that there exists no linear combination which is $I(0)$, i.e. $x_{1t}$ and $x_{2t}$ are not cointegrated (cf. Engle-Granger, 1987).

Let us now make the following assumption:

Assumption A2. (i) $x_{2t}$ does not Granger-cause at frequency 0, i.e. in the long-run, $x_{1t}$, (ii) $x_{1t}$ Granger-cause at frequency 0 $x_{2t}$.

Thus, we assume one-way causality in the long-run. Granger and Lin (1995) discuss in detail the notion of causality at different frequencies.\(^1\)

In order to show the main implications of non-causality in the long-run, let us start with the bivariate Wold representation (1). First, we normalize for convenience the variances\(^2\), so that

\(^1\) Building on previous works by Geweke (1982) and Hosoya (1991).
\(^2\) As Granger and Lin point out, the normalization does not affect the causal measure.
\[ \text{var}(\varepsilon_{1t}) = \text{var}(\varepsilon_{2t}) = 1 \quad \text{and} \quad \text{corr}(\varepsilon_{1t}, \varepsilon_{2t}) = \rho. \]  
Second, we rewrite (1) as:

\[ \Delta X_t = D(L)e_t \]

(2)

where: \( D(L) = C(L)T \) and \( e_t = T^{-1}e_t \). \( T \) is a transformation matrix such that: (i) \( \text{corr}(\varepsilon_{1t}, \varepsilon_{2t}) = 0; \) (ii) \( \text{var}(\varepsilon_{1t}) = 1; \) (iii) \( \text{var}(\varepsilon_{2t}) = (1 - \rho^2) \).

Hence, we recover the spectral density of \( \Delta x_{1t} \), indicated with \( g(\lambda) \), from (2):

\[ g(\lambda) = (2\pi)^{-1} \left( |D_{11}(z)|^2 + |D_{12}(z)|^2 (1 - \rho^2) \right) \]

(3)

where \( z = e^{i\lambda} \) and \( -\pi \leq \lambda \leq \pi \).

Let us now indicate with \( M(\lambda) \) the measure of causality of \( \Delta x_{2t} \) to \( \Delta x_{1t} \) at frequency \( \lambda \). Then, \( M(\lambda) \) is defined as:

\[ M(\lambda) = \log \left( \frac{g(\lambda)}{|D_{11}(z)|^2} \right) \]

(4)

It is easy to verify that, given this definition, \( M(0) = 0 \), i.e. \( x_{2t} \) does not in the long-run cause \( x_{1t} \) if \( D_{12}(1) = 0 \). Noting that \( D_{12}(1) = C_{12}(1) \), this implies \( C_{12}(1) = 0 \).

In the following part of this section we briefly review the main features of structural VAR models.

We can always think of (1) as a reduced-form relation of the following structural model:

\[ \Delta X_t = B(L)\eta_t \]

(5)

where: (i) \( B(L) = C(L)B(0) \); (ii) \( \eta_t = B(0)^{-1}\varepsilon_t \); (iii) \( E(\eta_t\eta_t') = I \).

\( \eta_t = (\eta_{1t}, \eta_{2t})' \) is a \((2 \times 1)\) vector of structural disturbances. Equation (1) and (2) imply that \( B(0)B(0)' = \Omega_e \). Hence, three restrictions are imposed on the four elements of \( B(0) \) and we
need one more restriction in order to obtain exact identification. In Blanchard and Quah (1989) identification is achieved by a long-run restriction: $\eta_{2t}$ has only a transitory effect on the level of $x_{1t}$. Thus, in this case, the fourth restriction is imposed on the matrix of long-run multipliers and the second column of $B(1)$ is such that $B_{12}(1) = 0$. This model provides identification of two structural shocks: the first, $\eta_{1t}$, has a permanent effect on the level of $x_{1t}$, whereas the second has no long-run effect on $x_{1t}$. It is worth noticing that unless we assume cointegration\(^3\), the second shock has a permanent effect on the level of $x_{2t}$.

These identifying restrictions stem from a keynesian model in which, due to nominal rigidity, aggregate demand shocks explain fluctuations in economic activity in the short-run. Nevertheless, the economic system converges to the steady-growth path in the long-run\(^4\).

It would be possible to achieve exact identification by imposing a Wold recursive form (cf. Sims, 1980), with $\Delta x_{1t}$ ordered first. In this case, it is assumed that $B(0)$ is lower triangular, i.e. the contemporary effect of $\eta_{2t}$ on $x_{1t}$ is restricted to zero. Hence, this orthogonalization of the residuals rests on the assumption that the underlying economic model has a contemporaneous recursive structure\(^5\).

In general, a recursive form does not identify a permanent and a transitory shock since the long-run effects of disturbances are not restricted. Nevertheless, the next section shows that, if A1 and A2 hold, then the triangular representation implies that a shock to the variable which is ordered second has no effect on the level of the variable first in the long-run.

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\(^3\) King, Plosser, Stock and Watson (1991) extended the Blanchard-Quah model to the cointegration case. However, it should be noted that the Blanchard-Quah model could itself be interpreted as a particular case of cointegrated system.

\(^4\) The restrictions imposed by Blanchard and Quah are based on a model by Fischer (1977).

\(^5\) "An assumption which is usually not motivated by the relevant economic theory". (Bernanke, 1986, p. 55)
3. The main result

In this section we will show that, if A1 and A2 hold, then by imposing a recursive VAR ordering we identify a permanent and a transitory shock for the variable which is ordered first. We will also show that Sims (1980) and Blanchard-Quah (1989) orthogonalizations are equivalent. Thus, in this case, the triangular representation provides a structural interpretation of the disturbances. The following proposition establishes the main result of this paper.

**Proposition 1.** Given the vector \( X_t = (x_{1t}, x_{2t})' \) let assumptions A1 and A2 be satisfied. Then, by imposing a recursive VAR ordering, with \( \Delta x_{1t} \) ordered first, one identifies a permanent and a transitory shock which affect \( x_{1t} \).

Proof. Let \( H(0) \) be the unique lower triangular matrix (Cholesky factor) such that \( H(0)H(0)' = \Omega \). Rewrite (1) as:

\[
\Delta X_t = H(L)\eta_t
\]

where: \( H(L) = C(L)H(0) \), \( \eta_t = H(0)^{-1}\varepsilon_t \), and \( E(\eta_t\eta_t') = I \).

Let us notice that \( H(1) = C(1)H(0) \). From A2 it follows that \( C(1) \) is lower triangular. Thus the second column of \( H(1) \) is such that \( H_{22}(1) = 0 \), i.e. \( \eta_{2t} \) has a transitory effect on \( x_{1t} \). Moreover, A1 implies that at least one of the two shocks has a permanent effect on the level of \( x_{1t} \).

Q.E.D.

**Remark 1.** Given assumptions A1 and A2 we identify, by imposing a recursive model, two structural shocks which affect \( x_{1t} \), and such that one is permanent and one is transitory. Thus, A1 and A2 imply that Sims (1980) orthogonalization, with \( \Delta x_{1t} \) first in the causal ordering, and Blanchard-Quah (1989) orthogonalization, with long-run restriction, are equivalent. Notice that the Blanchard-Quah representation can be obtained as an orthonormal transformation of \( H(0) \), i.e. by postmultiplying \( H(0) \) for a non-singular matrix \( M \) such that \( MM' = I \) and such that \( x_{1t} \) is affected by a permanent and by a transitory shock. Thus, in this case, the matrix \( M \) is the identity.
**Remark 2.** Assumption A2 ensures unidirectional causality at frequency 0. It is important to note that if the components of $X_t$ are not cointegrated, then the two series cannot cause each other in the long-run. However, it is not possible to rule out the case in which there is non-causality at frequency 0 in both directions. Hence, given a pair of integrated series, assumption A2 implies that a structural recursive VAR representation exists and is unique.

**Remark 3.** If we assume $C_{12}(L) = 0$, i.e. Granger non-causality at all frequencies, then, *a fortiori*, a shock to the variable which is ordered second has no effect on the level of the variable first in the long-run. Hence, Granger non-causality is sufficient but not necessary since it implies that there is no effect at all horizons.

### 4. Cointegrated systems

In this section we consider the case in which $x_{1t}$ and $x_{2t}$ are both I(1) but there exists a stationary linear combination of the two series, so that they are cointegrated. We will show that if assumption A2 holds, then a shock to $x_{2t}$, holding (contemporaneously) constant $x_{1t}$, has only transitory effects on the level of both variables.

Thus the following assumption holds:

**Assumption A3.** The vector $X_t$ has an Error-Correction Model (ECM) representation:

$$
\Delta X_t = \Gamma(L)\Delta X_{t-1} - A(1)X_{t-1} + \varepsilon_t
$$

where: (i) $A(1) = \alpha\beta'$ has reduced rank $r = 1$; (ii) $\alpha$ is a $(2 \times 1)$ vector of loadings and $\beta$ is a $(2 \times 1)$ vector of coefficients in the cointegrating vector. The reduced-form Wold representation of (5) is given by:

$$
\Delta X_t = C(L)\varepsilon_t
$$
where: (i) \( C(1) = \beta_\perp \gamma \alpha'_\perp \). \( \alpha'_\perp \) and \( \beta_\perp \) are, respectively, the orthogonal complements to the matrix of error correction coefficients and the matrix of cointegration vectors, i.e. \( \beta'_\perp \beta_\perp = 0, \alpha'_\perp \alpha_\perp = 0 \); (ii) \( \gamma = (\alpha'_\perp \Psi \beta_\perp)^{-1} \) where \( \Psi \) is the derivative of \( \Psi(z) \), the characteristic polynomial of model (5), for \( z = 1 \); (iii) the spectral density matrix of \( \Delta X_t \) evaluated at frequency 0, \( 2\pi^{-1} C(1) \Omega C(1)' \), has reduced rank \( r = 1 \).

Assumption A4. \( x_{2t} \) does not Granger-cause at frequency 0 \( x_{1t} \), i.e. \( \alpha'_\perp = (1, 0) \) (cf. Granger and Lin, 1995). It is worth noticing that assumption A4 implies \( x_{1t} \) does not adjust to the steady-state equilibrium.

We now state the following result.

Proposition 2. Given the vector \( X_t = (x_{1t}, x_{2t})' \) let assumptions A3 and A4 be satisfied. Then, a recursive VAR, with \( \Delta x_{1t} \) first in the causal ordering, is such that the first shock has a permanent effect on the level of both \( x_{1t} \) and \( x_{2t} \), whereas the second shock has only a transitory effect on the level of both variables.

Proof. Let \( H(0) \) be the unique lower triangular matrix such that \( H(0) H(0)' = \Omega \). Rewrite (6) as:

\[
\Delta X_t = H(L) \eta_t
\]

(7)

where: \( H(L) = C(L)H(0), \eta_t = H(0)^{-1} \varepsilon_t \) and \( E(\eta_t \eta_t') = I \).

Thus, \( H(1) = C(1)H(0) \) and, given assumption A4, it follows that the second column of \( C(1) \) is such that \( C_{12}(1) = C_{22}(1) = 0 \). But, since \( H(0) \) is lower triangular, the second column of \( H(1) \) also has zero elements, i.e. \( \eta_{2t} \) exhibits a transitory effect on both \( x_{1t} \) and \( x_{2t} \). Thus, only \( \eta_{1t} \) has a permanent effect on both variables.

Q.E.D.

Remark 5. Let us assume that the variable ordered first follows a random walk. Then, assumption A4 is trivially satisfied and shocks which affect the variable ordered second exhibit only transitory effects on both series. More precisely, as far as the variable which is first
is concerned, there is no effect at any horizon. This result was first shown by Cochrane (1994). Proposition 2 thus implies a generalization of Cochrane’s result: in order to identify by a recursive structure a permanent and a transitory shock, assuming a random walk process for the variable first is a sufficient but not necessary condition.

Remark 6. The bivariate model estimated by Blanchard and Quah (1989) can be interpreted as a cointegrated system. In fact, they assume that GNP (Y) is I(1), whereas the rate of unemployment (U) is I(0). Thus, the vector $Z = (Y, U)$ is cointegrated with cointegrating vector $(0, 1)'$. In a recent paper Crowder (1995) shows that the joint restriction $\beta = (0, 1)'$ and $\alpha'_{\perp} = (1, 0)$ is not rejected. The interesting conclusion is that, given the estimated model, the structural disturbances could have been recovered by imposing a recursive structure with $\Delta Y$ first in the causal ordering.

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6 Indeed, such a structural model was estimated by Evans (1989). "...However, instead of using the long-run restriction that we used here, he assumes that supply disturbances have no contemporaneous effect on output. We find this restriction less appealing as a way of achieving identification" (Blanchard-Quah, 1989, p.660). Undoubtedly, the Blanchard-Quah model represents an important point of departure with respect to the traditional identification scheme. Nevertheless, as far as their bivariate system is concerned, the two schemes are equivalent.
5. Conclusions and implications for empirical research

According to many researchers, the traditional Cholesky orthogonalization does not provide a structural interpretation of the disturbances which affect the economic system, unless the true economic model is recursive. Nevertheless, in this paper we have shown that given a bivariate VAR model which includes I(1) variables, if the two series do not cause each other at frequency 0, i.e. in the long-run, then a triangular representation identifies structural responses. Moreover, in this case, Sims (1980) orthogonalization, based on a recursive structure, and Blanchard-Quah (1989) orthogonalization, based on a long-run restriction, are equivalent. These results suggest that economic models which predict long-run neutrality have a natural recursive VAR interpretation. For example, let us consider output and inflation models. These models predict that: (1) given a vertical long-run aggregate supply curve in the output-inflation space, nominal shocks have no permanent effect on the level of output; (2) a short-run trade-off between inflation and output could arise, due to nominal rigidities (keynesian approach) or expectation errors (competitive equilibrium approach). These models thus lead to the following structural VAR representation: a shock to the rate of inflation, holding contemporaneously constant output, has only a transitory effect on output. In other words, we argue that these models predict non-causality at frequency 0 of the rate of inflation to output.

In recent years, the real-business-cycle approach to economic fluctuations has challenged both the keynesian and expectation errors view (e.g. Prescott, 1986). The real-business-cycle models predict that: (1) movements in economic activity are dominated by technology shocks; (2) nominal side disturbances do not affect output at any horizon. Moreover, both real and monetary shocks have effects on inflation. The interesting conclusion is that, a fortiori, these models have a structural recursive VAR interpretation, since they establish unidirectional causality at all frequencies.

Our results are thus consistent with the idea that the Cholesky orthogonalization is not an atheoretical identification. We argue, however, that a wide range of economic models can be associated with contemporaneous recursive structures.
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