A Model for Pricing an Option with a Fuzzy Payoff

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September 1999

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ABSTRACT

The aim of this paper is to price an option in a one-period model when the price of the underlying asset is vague. The vagueness is modelled by the use of triangular fuzzy numbers and the pricing methodology is based on the no arbitrage principle. This work extends the binomial option-pricing model to include different levels of information.

Keywords: pricing, options, Fuzzy sets.
JEL classification: G13, G14.

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§ The authors gratefully acknowledge support from CNR and MURST.
1. INTRODUCTION

The aim of this paper is to provide an alternative framework to option pricing based on fuzzy set theory, which offers a more intuitive and simple way of modelling uncertainty. The pricing methodology is still the no arbitrage principle. Yet, the price of the underlying is fuzzy.

The plan of the paper is the following. In section 2 we use fuzzy set theory to model the uncertainty in the price of a vague asset. In section 3 we set out the foundations and the assumptions of the model. In section 4 we present the characteristics of an option on a fuzzy asset and in the following section we describe the pricing methodology used. In section 6 we analyse the main factors affecting the call option price. In section 7 we discuss the relationship between our model and the standard Binomial Option Pricing Model. The last section provides conclusions and some lines of future research work.

2. MODELING THE PRICE OF A FUZZY ASSET

In this section we describe the vagueness of the future price of an asset with uncertain payoffs, with reference to a one period model, whereby time \( t \in \{0,1\} \). Let \( X \) be the real line, representing a set of monetary values and let \( F \) be the set of all fuzzy subsets of \( X \). Let \( N \in F \) be the set of all the convex and normal subsets, i.e. of all fuzzy numbers. Let \( R \) be the fuzzy asset and \( R^R \) its price at time 1. As \( R \) is fuzzy, \( R^R \) is unknown. We assume that the representative investor can only guess that the price will be included in a given interval. Specifically, he may think that \( R^R \) will neither exceed an upper bound, call it \( a_3 \), nor a lower bound, call it \( a_1 \), and inside this interval, suppose that the best guess is a given value called \( a_2 \). In order to represent this idea, we assume for simplicity \( R^R \) as a triangular fuzzy number identified by its characteristic function \( \mu_{R^R(x)} \):
where $x \in X$, $0 \leq a_1 \leq a_2 \leq a_3$.

More simply we can write: $p_1^R = (a_1, a_2, a_3)$, where $a_1$ is the minimum possible value, $a_3$ the maximum, and $a_2$ the most possible. In fact, the fuzzy set $p_1^R$ induces a possibility distribution on the value of $x$, that is equal to its characteristic function. See for example Fig.1.

Alternatively, we can write a triangular fuzzy number in terms of its $\alpha$-cuts (or confidence intervals) (see Fig.1) by the following formula:

$$p_1^R(\alpha) = [a_1(\alpha), a_3(\alpha)] = [a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)]$$

where $\alpha$ is the level of confidence, $\alpha \in [0,1]$.

This representation will be useful to do some algebra with fuzzy numbers.

![Figure 1. The price of the underlying at $t=1$.](image)
3. PRELIMINARIES ON THE MODEL

Let us consider a one-period model where $t \in [0,1]$ is time; $W$ is the set of traded portfolios; if $w \in W$, $P^w_t$ be the value at time $t$, of asset $w$. The two basic securities are $M, R \in W$: the first, $M$, called money market account, is riskless, while the second $R$, is the fuzzy asset. The money market account, is worth one at $t=0$, $P^M_0 = 1$ and its value at $t=1$ is $P^M_1 = 1 + r$, where $r$ is the risk-free interest rate. The second security $R$, is fuzzy: the price at time zero, $P^R_0$, is observable, while the price at time one, $P^R_1$, is modelled as the fuzzy triangular number $P^R_1 = (a_1, a_2, a_3)$ introduced in the previous section.

We also make the following assumptions:

A1) $W$ is a linear space. If assets $i, j \in W$, and $\alpha, \beta \in \mathbb{R}$, then $\alpha P^i_t + \beta P^j_t \in W$, i.e. marketed assets are closed under the construction of portfolios.

A2) All investors have homogeneous beliefs for each $\alpha \in [0,1]$. Every agent agree that the price at time $t$ of asset $w$ will be $P^w_t$.

A3) Markets are frictionless i.e. markets have no transaction costs, no taxes, no restrictions on short sales and asset are infinitely divisible.

A4) Every investor acts as a price taker.

A5) Interest rates are positive. The interest rate is equal to $r > 0$ percent per unit time.

A6) The fuzzy asset price at $t=1$ is represented by a triangular fuzzy number as introduced in section 2.

A7) No arbitrage opportunities are allowed. This condition is expressed by the following formula:

$$a_1(\alpha) \leq P^R_0 (1+r) \leq a_3(\alpha) \quad \forall \alpha \in [0,1]$$

Were it not verified, the fuzzy asset would have a price strictly greater, or less than the price of the money market account, and arbitrage opportunities would arise.

A8) The market is complete $\forall \alpha \in [0,1]$. The condition for completeness is $\alpha \neq 1$, in fact if $\alpha = 1$, $P^R_1 = a_3$, and the asset becomes crisp (i.e. not fuzzy), and there is no more uncertainty in the model.
4. CHARACTERISTICS OF AN OPTION ON A FUZZY ASSET

An option is a bilateral contract which gives the holder the right to buy (call option) or to sell (put option) a certain asset (the underlying) at a given price $K$ (strike price), at the maturity date $T$. As we are in a one-period model, it makes no sense to distinguish between European and American ones. We examine in detail the case of a call option, the case of a put is symmetrical and will not be discussed in this paper.

At the maturity date, a call option has a positive value if the price of the underlying is greater than the exercise price; in the opposite case it remains unexercised and has zero value. As the payoff of a call option depends on the price of the underlying asset, if the latter is fuzzy, the former is fuzzy too.

Let $K$ be the strike price, in order to make an option an interesting contract we assume that

$$a_1(\alpha) \leq K \leq a_3(\alpha) \quad \forall \alpha \in [0,1] \quad (1)$$

The payoff of the call at $t=1$ is:

$$P^C_t = \max[0, (P^S_t - K)] \quad (2)$$

and, since $P^S_t$ is a triangular fuzzy number,

$$P^C_t = \max[0, (a_1 - K, a_2 - K, a_3 - K)] \quad (3)$$

In order to fulfill condition A.7) and equation (1) the payoff of the call is given only for

$$\alpha < \bar{\alpha}, \quad (4)$$

where

$$\bar{\alpha} = \min[\alpha^*(K), \alpha^*(P^S_t(1+r)), \alpha^*(P^S_t(1+r))]$$

$$\alpha^*(K) = \frac{K - a_1}{a_2 - a_1} \quad \alpha^*(P^S_t(1+r)) = \frac{P^S_t(1+r) - a_1}{a_2 - a_1}$$

$$\alpha^*(K) = \frac{a_3 - K}{a_3 - a_2} \quad \alpha^*(P^S_t(1+r)) = \frac{a_3 - P^S_t(1+r)}{a_3 - a_2}$$

Thus, the payoff of the call at $t=1$ is:

$$P^C_t = \max\{0, a_3 - K - \alpha (a_3 - a_2)\} \quad \alpha < \bar{\alpha} \quad (5)$$

Note that the value of the option is not a triangular number, a result that depends on the nonlinear nature of option contracts. An example is provided in Fig.2.
5. PRICING AN OPTION ON A FUZZY ASSET

In this section we show how to price the call option by constructing a portfolio that replicates the payoff of the call at time one. Then, by the no arbitrage condition, the price of the call at time zero must be equal to the price of the portfolio at the same time.

Let $N_i$ be the number of units of asset $i$, with $i = M, R$, we look for a portfolio such that:

$$N_M p^M + N_R p^R = p^C$$

Thus, for each $\alpha$, we get the following system:

$$N_M (1+r) + N_R [a_1 + \alpha(a_2 - a_1)] = 0$$
$$N_M (1+r) + N_R [a_3 - \alpha(a_3 - a_1)] = a_3 - \alpha(a_3 - a_1)$$

the solution is the following:

$$N_M = \frac{N_R [a_1 + \alpha(a_2 - a_1)]}{(1+r)}$$
$$N_R = \frac{a_3 (1-\alpha) - K + a_2 \alpha}{(1-\alpha)(a_3 - a_1)}$$

Figure 2. An example of the payoff of the call (dashed area).
thus the price of the call is:
\[ P_0^c = N_M P_0^M + N_R P_0^R \]  
(8)

Recall that the price of the Call is defined only for \( \alpha < \bar{\alpha} \).

Alternatively, we can price the call by means of the risk neutral valuation approach, as follows:
\[ P_0^c = \frac{1}{1 + r} \hat{E}[P_1^c] \]

where \( \hat{E} \) stands for expectation under the following risk neutral probabilities that we derive for each \( \alpha < \bar{\alpha} \):
\[
\begin{align*}
q_1 &= \frac{(1 + r) - d - \alpha (m - d)}{(u - d)(1 - \alpha)} \\
q_2 &= \frac{(u - d)(1 - \alpha) - (1 + r) + d + \alpha (m - d)}{(u - d)(1 - \alpha)}
\end{align*}
\]
(9)

where \( u = \frac{a_1}{P_0^R}, \quad m = \frac{a_2}{P_0^R}, \quad d = \frac{a_3}{P_0^R} \).

6. PROPERTIES OF THE CALL PRICE

In order to assess the validity of this model, we verify whether the standard properties of option prices hold.

Recall that the following conditions are always verified:
\[ 0 \leq a_1 \leq a_2 \leq a_3 \]  
(10)
\[ \alpha \in [0,1[ \]  
(11)

Rewriting equation (8) we can write the price of the call as follows:
\[ P_0^c = \frac{a_3 (1 - \alpha) - K + a_2 \alpha}{(1 - \alpha)(a_3 - a_1)} \cdot \frac{P_0^R (1 + r) - a_1 - \alpha (a_2 - a_1)}{(1 + r)} \]  
(12)

The call price depends on the value of the parameters \( K, r, P_0^R, \alpha \) and on the shape of the distribution, i.e. \( a_1, a_3, a_2 \).

In the following we will examine separately the influence of each parameter.

Deriving equation (12) with respect to the strike price \( K \), we obtain:
Given conditions (10), (11) and assumption A5 it is easy to see that the denominator is always positive and the numerator is positive if:

\[
\alpha < \frac{p^R_0(1+r) - a_1}{(a_2 - a_1)}
\]

and this is always verified, given condition (4). Thus, the price of the call is decreasing in the strike price.

Deriving equation (12) with respect to \( r \), the interest rate, we obtain:

\[
\frac{\partial P^C_0}{\partial r} = \left[ a_3(1-\alpha) - K + a_2\alpha \right] \cdot \frac{a_1 + \alpha(a_2 - a_1)}{(1-\alpha)(a_3 - a_1)} \cdot \frac{1}{(1+r)^2}
\]

The second term in brackets and the denominator of the first are always positive, given conditions (10) and (11), the numerator of the first term is positive if:

\[
\alpha < \frac{a_3 - K}{(a_3 - a_2)}
\]

and this is always verified, given condition (4). Thus the price of the call is increasing in \( r \).

As \( r \) increases, the expected growth in the stock price increases and this implies an increase in the value of the call, however, as \( r \) increases, the present value of any future cash flow received by the holder of the option decreases. It can be shown that the first effect dominates the second.

Deriving equation (12) with respect to \( P^R_0 \), the price of the underlying, we obtain:

\[
\frac{\partial P^C_0}{\partial P^R_0} = \left[ a_3(1-\alpha) - K + a_2\alpha \right] \cdot \frac{a_1 + \alpha(a_2 - a_1)}{(1-\alpha)(a_3 - a_1)}
\]

The denominator is always positive, given conditions (10) and (11) and the numerator is positive if:

\[
\alpha < \frac{a_3 - K}{(a_3 - a_2)}
\]

and this is always verified, given condition (4). Thus the price of the call is increasing in \( P^R_0 \).

Summing up, the call price in our model has the standard properties of call option prices with respect to the strike price, the interest rate and the underlying price.
We further examine the additional properties of the call price that are specific to our model.

Deriving equation (12) with respect to $\alpha$ we obtain:

\[
\frac{\partial P_c}{\partial \alpha} = \left[ \frac{(2\alpha-a^2)(a_3-a_2)(a_2-a_1)+(a_2-K)(P_0^r(1+r)) + a_1(a_2-a_2)+a_2(K-a_3)}{(1-\alpha)^2(a_3-a_1)(1+r)} \right]
\] (16)

The denominator is always positive, given condition (10) and assumption A5. It is easy to prove that the numerator is negative for each $\alpha$ that satisfies condition (4) and thus the derivative with respect to $\alpha$ is always negative. As $\alpha$ may be interpreted as the information level, this result is in line with the standard properties of a call option, given that the lack of information about the underlying causes an increase in the volatility.

Deriving equation (12) with respect to $a_1$, the lower bound of the fuzzy number, we obtain:

\[
\frac{\partial P_c}{\partial a_1} = \left[ \frac{a_3(1-\alpha) - K + a_3\alpha}{(1-\alpha)(1+r)} \right] * \left[ \frac{P_0^R(1+r) - a_3(1-\alpha) - a_3\alpha}{(a_3-a_1)^2} \right]
\] (17)

Given condition (10) and assumption A5), the first term in parenthesis is positive for $\alpha < \alpha^{**}(K)$; the second term is positive for $\alpha > \alpha^{**}(P_0^r(1+r))$, therefore it is easy to show that for $\alpha < \bar{\alpha}$ the derivative is always negative.

Deriving equation (12) with respect to $a_3$, the upper bound of the fuzzy number, we obtain:

\[
\frac{\partial P_c}{\partial a_3} = \left[ \frac{P_0^R(1+r) - a_1 - a_3}{(1-\alpha)(1+r)} \right] * \left[ \frac{K - a_3 - a_3\alpha}{(a_3-a_1)^2} \right]
\] (18)

Given condition (11) and assumption A5), the first term in parenthesis is positive for $\alpha < \alpha^{*}(K)$; the second term is positive for $\alpha > \alpha^{*}(P_0^r(1+r))$, therefore it is easy to show that the derivative is always positive.

These two latter results can be easily interpreted by noting that keeping all the other parameters fixed, an increase in $a_1$ or a decrease in $a_3$ is equivalent to a reduction in the volatility of the underlying, which obviously implies a decrease in the call value.

Deriving equation (12) with respect to $a_2$, the peak value of the fuzzy number, we obtain:

\[
\frac{\partial P_c}{\partial a_2} = \left[ \frac{\alpha'[P_0^R(1+r) - a_1 - a_3 + K - \alpha(2a_2-a_1-a_3)]}{(1-\alpha)(1+r)(a_3-a_1)} \right]
\] (19)
Since the denominator is always positive, given conditions (10), (11) and assumption A5), the derivative is equal to zero if $\alpha=0$ or if the quantity in square brackets at the numerator is equal to zero, i.e. if

$$\alpha = \tilde{\alpha}$$

where $\tilde{\alpha} = \frac{P_0^R (1 + r) - a_1 - a_3 + K}{2a_2 - a_1 - a_3}$. 

Thus, the sign of the derivative depends on the values of the parameters $K$, $P_0^R$, $a_1$, $a_2$, $a_3$.

A summary of the different possibilities emerging is given in Table 1. As we can see the interpretation of these results is not straightforward since it strongly depends on the relative magnitude of the model parameters.

<table>
<thead>
<tr>
<th>$a_2-a_1=a_3-a_2$</th>
<th>$a_2-a_1&gt;a_3-a_2$</th>
<th>$a_2-a_1&lt;a_3-a_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_0^R (1 + r) - a_1 = a_3 - K$</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>$P_0^R (1 + r) - a_1 &gt; a_3 - K$</td>
<td>+</td>
<td>{ + if $\alpha &lt; \tilde{\alpha}$ }</td>
</tr>
<tr>
<td>{ - if $\alpha &gt; \tilde{\alpha}$ }</td>
<td>{ 0 if $\alpha = \tilde{\alpha}$ }</td>
<td></td>
</tr>
<tr>
<td>$P_0^R (1 + r) - a_1 &lt; a_3 - K$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>{ + if $\alpha &gt; \tilde{\alpha}$ }</td>
<td>{ 0 if $\alpha = \tilde{\alpha}$ }</td>
<td></td>
</tr>
</tbody>
</table>

Table 1. The sign of the derivative of the call price with respect to the peak value of the underlying.
7. COMPARISON WITH THE BINOMIAL MODEL

In order to facilitate the comparison, we rewrite the one-period Binomial Option Pricing Model in the following way. Let \( t=\{0\} \) be time; \( W \) be the set of the marketed portfolios; \( S=\{D,U\} \) be the set of the states of the world, where \( U \) means up, \( D \) down; \( K \) the exercise price, with \( a_1 < K < a_2 \). If \( w \in W \), \( P_t^w(S) \) is the value at time \( t \), of asset \( w \) in state \( S \). Let \( M \) be a risk free asset, with \( P_{1M}^w = 1, P_{1M}^U = 1 + r \) in both states of the world. Let \( R \) be a risky asset with a given price \( P_0^R \), whose future price depends on the state of the world: \( P_1^R(D) = a_1, P_1^R(U) = a_3 - K \). Pricing the option by the construction of the duplicating portfolio gives:

\[
N_M = \frac{a_1(a_3 - K)}{(1+r)(a_3 - a_1)} \\
N_R = \frac{(a_3 - K)(a_3 - a_1)}{(a_3 - a_1)}
\]

where \( N_w \) is the quantity of asset \( w \).

These are the same values that we found in our model if \( \alpha = 0 \) (see system (7)).

Alternatively, the standard binomial model can be represented by means of the risk neutral valuation approach, as follows:

\[
P_v^C = \frac{1}{1+r} \hat{E} [P_1^C]
\]

where \( \hat{E} \) stands for expectation under the following risk neutral probabilities:

\[
\begin{align*}
q_1 &= \frac{(1+r)-d}{(u-d)} \\
q_2 &= \frac{u-(1+r)}{(u-d)}
\end{align*}
\]

Note that these values are the same that we found in our model if \( \alpha = 0 \) (see system (9)).
7. CONCLUSIONS

As far as we know, this is the first attempt of combining a standard Binomial Option Pricing Model with a fuzzy representation of the option payoff. In this paper we have proposed, within a one period framework, a fuzzy modelling of the uncertainty in the price of an asset and we have priced, by the no arbitrage condition, an option written on that asset. This methodology offers some advantages. First, it provides a more realistic way of looking at the future price of a fuzzy asset. Second, it includes the results of the Standard Binomial Model. This work has to be seen as preliminary to future research and still lends itself to be extended in many directions. High on the research agenda are the extension to a multiperiod discrete version and the development of a model with informational asymmetries, allowing for different \( \alpha \), i.e. the existence of differently informed agents in the market.

REFERENCES

60. Andrea Giovannetti [1990] "La probabilità individuale di risposta nel trattamento dei dati mancanti". pp. 13
63. Andrea Ginzburg [1990] "Debito pubblico, teorie montane e tradizione civica nell’Inghilterra del Settecento". pp. 30
64. Mario Forni [1990] "Incertezza, informazione e mercati assicurativi: una rassegna". pp. 37
67. Paola Bertolini [1990] "La situazione agro-alimentare nei paesi ad economia avanzata". pp. 20
70. Margherita Russo [1990] "Cambiamento tecnico e distretto industriale: una verifica empirica". pp. 113
73. Rita Patrinieri [1990] "La populazione italiana: problemi di oggi e di domani". pp. 57
74. Enrico Giovannetti [1990] "Illusioni ottiche negli andamenti delle Grandezze distributive: la scala mobile e l”appiattimento” delle retribuzioni in una ricerca". pp. 120
77. Antonietta Bassetti e Costanza Torricelli [1990] "Una riqualificazione dell”approco bargaining alla selezione di portafogli". pp. 4
78. Antonietta Bassetti e Costanza Torricelli [1990] "Il portafoglio ottimo come soluzione di un gioco bargaining". pp. 15
79. Mario Forni [1990] "Una nota sull”errore di aggregazione". pp. 6
80. Francesca Bergamini [1991] "Alcune considerazioni sulle soluzioni di un gioco bargaining". pp. 21
84. Sebastiano Brusco e Sergio Paba [1991] "Connessioni, competenze e capacità concorrenziale nell”industria della Sardegna". pp. 25
89. Maria Cristina Marozzio [1992] "La relazione salari-occupazione tra rigidità reale e rigidità nominale". pp. 30
90. Mario Biagioli [1992] "Employee financial participation in enterprise results in Italy". pp. 50
91. Mario Biagioli [1992] "Wage structure, relative prices and international competitiveness". pp. 50
96. Paolo Emilio Mustrilli [1993] "Debito pubblico, intermediari finanziari e tassi d”interesse: il caso italiano". pp. 30
99. Marcello D’Amato e Barbara Pistoresi [1994] "The relationship(s) among Wages, Prisons, Unemployment and Productivity in Italy". pp. 30
101. Barbara Pistoresi [1994] "Using a VECM to characterise the relative importance of permanent and transitory components". pp. 28
102. Gian Paolo Caselli e Gabriele Pastrello [1994] "Polish recovery form the slump to an old dilemma". pp. 20
103. Sergio Paba [1994] "Imprese visibili, accesso al mercato e organizzazione della produzione". pp. 20
105. Giuseppe Marotta [1994] "Credit view and trade credit: evidence from Italy". pp. 20
106. Margherita Russo [1994] "Unit of investigation for local economic development policies". pp. 25
167. Marcello D’Amato e Barbara Pistorese [1996] “So many Italians: Statistical Evidence on Regional Cohesion” pp. 31
173. Mauro Dell’Amico [1997] “A Linear Time Algorithm for Scheduling Outforests with Communication Delays on Two or Three Processors” pp. 18
175. Paolo Boni e Massimo Matteuzzi [1997] “Nuovi strumenti per l’assistenza social” pp. 31
176. Mauro Dell’Amico, Francesco Maffioli e Marco Trubian [1997] “New bounds for optimum traffic assignment in satellite communications” pp. 21
185. Gian Paolo Castelli e Maurizio Battini [1997] “Following the tracks of allaisianu and mecklewright the changing distribution of income and earnings in poland from 1989 to 1995” pp. 21
186. Mauro Dell’Amico e Francesco Maffioli [1997] “Combining Linear and Non-Linear Objectives in Spanning Tree Problems” pp. 21
203. Stefano Bordoni [1997] “Supporto Informatico per la Ricerca delle soluzioni di Problemi Decisionali” pp. 30
212. Alberto Rovereto [1997] “Asymptotic prior to posterior analysis for graphical gaussian models” pp. 8
214. Gian Paolo Castelli e Franca Manghi [1997] “La transizione da piano a mercato e il modello di liang” pp. 15
229. Tommaso Minerva e Irene Poli [1998] “Building an ARMA Model by using a Genetic Algorithm” pp. 60
232. Alberto Roverato e Irene Poli [1998] “Un algoritmo genetico per la selezione di modelli grafico” pp 11
271 Antonella Picchio [1999] "La questione del lavoro non pagato nella produzione di servizi nel nucleo domestico (Household)" pp.58.