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ABSTRACT: This paper characterizes the Weak Weak Axiom of Revealed Preference (WWA) and Wald's Weak Axiom (WALD). The idea underlying this work is that the Weak Axioms can be reformulated in terms of monotonicity properties of demand functions, so that their differential characterization can be obtained by standard mathematical thechniques. The differential characterization of WWA was first derived by Kihlstrom, Mas-Colell and Sonnenschein (1976), and then Hildenbrand and Jerison (1989). The differential characterization of WALD is a relatively recent result and was obtained by John (1991) and Hildenbrand (1994). In this paper we shall derive both these characterizations by using an alternative method of proof which is essentially based on two simple lemmas.

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MONOTONICITY AND THE DEMAND THEORY OF THE WEAK AXIOMS

INTRODUCTION

This paper deals with essentially two properties of demand functions, the Weak Weak Axiom of Revealed Preference (WWA) and Wald's Weak Axiom (WALD). WWA is a milder version of Samuelson's Weak Axiom of Revealed Preference and was first introduced by Hicks (1956). Similarly, WALD is a milder version of a condition on aggregate excess demand functions proposed by A. Wald in the context of general equilibrium theory. The relevance of these two concepts is not confined to the theory of individual choice but also extends to equilibrium analysis. Although WWA and WALD are among the weakest conditions of consistency of consumer behaviour, when possessed by market demand they may turn out to be strong enough to ensure uniqueness of equilibrium.

Monotonicity of the demand function is a property with a direct economic interpretation; it amounts to the 'Law of Demand' and implies that the demand of every good is decreasing with respect to its own price. From the analytical point of view monotonic functions are characterized by the negative semi definiteness of their jacobian, therefore one immediately obtains a complete differential characterization of the 'Law of Demand'. The idea underlying this paper is that the weak axioms can be reformulated in terms of monotonicity properties of demand functions so that their differential characterization can be obtained by standard mathematical techniques.

First, we shall provide a characterization of WWA and WALD in terms of monotonicity properties of demand functions, called respectively GLD and RM; in particular, it is shown that WWA amounts to the monotonicity of the Slutsky compensated demand function, whereas WALD is equivalent to monotonicity of the demand function on a restricted price set. For homogeneous demand functions the two axioms coincide so that we obtain a result which is the finite analog of an equivalence result derived by Kihlstrom, Mas-Colell and Sonneschein (1976) and Hildenbrand and Jerison (1989); moreover, by means of a simple numerical example we shall show that the result cannot be strengthened. Secondly, we shall derive a complete differential characterization of WWA and WALD. The differential characterization of WWA was first derived by Kihlstrom, Mas-Colell and Sonnenschein (1976), and then Hildenbrand and Jerison (1989) simplified the method of proof and added other contributions. The differential characterization of WALD is a relatively recent result and was obtained by John (1991) and Hildenbrand (1993). In this paper we shall derive both these characterizations by using an alternative method of proof which is essentially based on two simple lemmas.

At the same time, this paper offers a somewhat more general approach to the demand theory of the weak axioms which allow to obtain simple proofs of other results found in this literature.

In the next section the basic notation and definitions are introduced. In section 2 the equivalencies between weak axioms and monotonicity properties are established; section 3 discusses the relation between weak axioms and homogeneity of demand functions; section 4 provides the differential characterization of the weak axioms by exploiting their relationship with monotonicity properties; section 5 summarizes the all discussion.

1. Notation and Definitions

A demand function $\boldsymbol{f}(\boldsymbol{p}, w)$ is defined as a continuously differentiable function $\boldsymbol{f}: \mathbb{R}^{\ell}_{++} \times \mathbb{R}_{++} \to \mathbb{R}^{\ell}_{+}$, satisfying *budget constraint*, i.e. $\boldsymbol{p} \cdot \boldsymbol{f}(\boldsymbol{p}, w) = w$. A demand function is *homogeneous* if $\boldsymbol{f}(t\boldsymbol{p}, tw) = \boldsymbol{f}(\boldsymbol{p}, w)$ for $t > 0$. The set of demand functions is denoted by $\mathcal{F}$, the subset of homogeneous demand functions by $\mathcal{F}_h$. An analogous notation is introduced for continuously differentiable demand functions, i.e. respectively C and C_h. Let us introduce the definition of the weak axioms:

Definition 1. A demand function is said to satisfy property:
WWA (Weak Weak Axiom) if for all $(\boldsymbol{p}, w)$ and $(\boldsymbol{q}, w')$

$$\boldsymbol{q} \cdot \boldsymbol{f}(\boldsymbol{p}, w) \leq w' \qquad \Longrightarrow \qquad \boldsymbol{p} \cdot \boldsymbol{f}(\boldsymbol{q}, w') \geq w.$$

WALD (Wald's Weak Axiom) if for all $\boldsymbol{p}$, $\boldsymbol{q}$ and w,

$$\boldsymbol{q} \cdot \boldsymbol{f}(\boldsymbol{p}, w) \leq w \qquad \Longrightarrow \qquad \boldsymbol{p} \cdot \boldsymbol{f}(\boldsymbol{q}, w) \geq w$$

or equivalently if[1]

$$(\boldsymbol{q} - \boldsymbol{p}) \cdot \boldsymbol{f}(\boldsymbol{p}, w) \leq 0 \qquad \Longrightarrow \qquad (\boldsymbol{q} - \boldsymbol{p}) \cdot \boldsymbol{f}(\boldsymbol{q}, w) \leq 0.$$

[1] This condition is known in the mathematical literature as 'pseudo monotonicity', see for example Karamardian, Schaible and Crouzeix (1993).

Clearly, WALD is the fixed income version of WWA; from the definitions it follows immediately that when demand functions are homogeneous the two axioms coincide. In section 3 we shall show that this is not true in general.

The Slutsky *compensated* demand function at prices $\boldsymbol{q}$ and relative to point $(\boldsymbol{p}, w)$ is defined by

$$\boldsymbol{s}(\boldsymbol{q}) = \boldsymbol{f}(\boldsymbol{q}, \boldsymbol{q} \cdot \boldsymbol{f}(\boldsymbol{p}, w));$$

the vector $\boldsymbol{s}(\boldsymbol{q})$ is the consumption bundle that the consumer would demand if the prices changed from $\boldsymbol{p}$ to $\boldsymbol{q}$ and his nominal income were compensated so as to keep unchanged his 'real income'. The compensated income is $\boldsymbol{q} \cdot \boldsymbol{s}(\boldsymbol{q}) = \boldsymbol{q} \cdot \boldsymbol{f}(\boldsymbol{p}, w)$ and recall that $\boldsymbol{s}(\boldsymbol{p}) = \boldsymbol{f}(\boldsymbol{p}, w)$. Let us now introduce the definitions of monotonicity that are needed in the sequel.

DEFINITION 2. A demand function is said to satisfy property:
GLD (Generalized Law of Demand) if the *compensated* demand function at point $(\boldsymbol{p}, w)$ is monotone, i.e.

$$(\boldsymbol{q} - \boldsymbol{p}) \cdot \big[\boldsymbol{s}(\boldsymbol{q}) - \boldsymbol{s}(\boldsymbol{p})\big] \leq 0,$$

for all $\boldsymbol{q}$ and all $(\boldsymbol{p}, w)$, or equivalently[2]

$$\boldsymbol{p} \cdot \big[\boldsymbol{s}(\boldsymbol{q}) - \boldsymbol{f}(\boldsymbol{p}, w)\big] \geq 0.$$

RM (Restricted Monotonicity) if the *demand function* is monotone on the set of prices $P_f = \{\boldsymbol{q} \in P \mid \boldsymbol{q} \cdot \boldsymbol{f}(\boldsymbol{p}, w) = w\}$, i.e.

$$\boldsymbol{p} \cdot \big[\boldsymbol{f}(\boldsymbol{q}, w) - \boldsymbol{f}(\boldsymbol{p}, w)\big] \geq 0,$$

for all $\boldsymbol{p}$ and w and for all $\boldsymbol{q} \in P_f$.[3] Equivalently RM can be defined as

$$(\boldsymbol{q} - \boldsymbol{p}) \cdot \boldsymbol{f}(\boldsymbol{p}, w) = 0 \qquad \Longrightarrow \qquad (\boldsymbol{q} - \boldsymbol{p}) \cdot \boldsymbol{f}(\boldsymbol{q}, w) \leq 0.$$

The economic interpretation of GLD is clear; it amounts to the 'Law of Demand' for compensated demand, which implies that, once compensated in terms of income, an agent cannot respond to an increase of the price of one good by rising its demand.

[2] We have used the fact that, by construction, $\boldsymbol{q} \cdot [\boldsymbol{s}(\boldsymbol{q}) - \boldsymbol{f}(\boldsymbol{p}, w)] = 0$.

[3] Clearly we have exploited the fact that $\boldsymbol{q} \cdot [\boldsymbol{f}(\boldsymbol{q}, w) - \boldsymbol{f}(\boldsymbol{p}, w)] = 0$ for $\boldsymbol{q} \in P_f$.

Similarly, Restricted Monotonicity (RM) can be interpreted in terms of a different kind of compensated demand. Whereas for the Slutsky compensated demand the consumer is compensated in terms of income for the loss of purchasing power induced by price variation, one can think of a different compensating mechanism which keeps nominal income constant and renormalizes the changed prices so that the consumer can still afford the initially chosen bundle. Then RM amounts to imposing that this compensated demand satisfies the 'Law of demand'.[4]

Let us close this section with a final set of definitions.

DEFINITION 3. A continuously differentiable demand function $\boldsymbol{f}(\boldsymbol{p}, w)$ is said to satisfy property
NSD: if the matrix of substitution terms, $S(\boldsymbol{p}, w)$, i.e. the Jacobian of the compensated demand at point $(\boldsymbol{p}, w)$, evaluated at $\boldsymbol{q} = \boldsymbol{p}$, is negative semi-definite, i.e.

$$\boldsymbol{v} \cdot S(\boldsymbol{p}, w) \cdot \boldsymbol{v} \leq 0$$

for all $\boldsymbol{v} \in \mathbb{R}^{\ell}$.
∂RM: if

$$\boldsymbol{v} \cdot \boldsymbol{f}(\boldsymbol{p}, w) = 0 \qquad \Longrightarrow \qquad \boldsymbol{v} \cdot \partial_p \boldsymbol{f}(\boldsymbol{p}, w) \boldsymbol{v} \leq 0$$

where $\boldsymbol{v} \in \mathbb{R}^{\ell}$.

It is quite intuitive that properties GLD and RM are respectively the finite counterparts of properties NSD and ∂RM. In section 4 we shall show that this is actually the case. The next section provides the characterization of the weak axioms in terms of the monotonicity properties given in Definition 2.

2. Monotonicity and the Weak Axioms

In this section we shall show, by means of a very simple argument, that WWA is equivalent to GLD and that WALD is equivalent to RM. Both of these results are obtained with same method of proof which is essentially based on the following Lemma.

[4] Clearly, when demand function are homogeneous the two concepts of monotonicity coincide.

LEMMA 1. *Let $\boldsymbol{f}(\boldsymbol{p},w)$ be a demand function and $\boldsymbol{p}$, $\boldsymbol{q} \in P$. If $\boldsymbol{q}\cdot\boldsymbol{f}(\boldsymbol{p},w) \leq w'$, then there exists $t^* \in (0,1]$ such that $\boldsymbol{q}\cdot\boldsymbol{f}(t^*\boldsymbol{p},w) = w'$.*

Proof. (i) Consider the function $h(t) = \boldsymbol{q}\cdot\boldsymbol{f}(t\boldsymbol{p},w)$ for $t \in (0,1]$. From the above assumption we have $h(1) \leq w'$. We want to show that there exists $t^* \in (0,1]$ such that $h(t^*) = \boldsymbol{q}\cdot\boldsymbol{f}(t^*\boldsymbol{p},w) = w'$. First, notice that from continuity of the demand function $h(t)$ is continuous. Second, by the budget constraint,

$$\boldsymbol{p}\cdot\boldsymbol{f}(t\boldsymbol{p},w) = \frac{w}{t};$$

since the members of the above expression tends to infinity as t goes to zero, at least one of the components of $\boldsymbol{f}(t\boldsymbol{p},w)$ goes to infinity as t goes to zero, which means that $h(t) \to \infty$ as $t \to 0$, since $\boldsymbol{q} \gg \boldsymbol{0}$.

■

Let us now state the main result of this section.

THEOREM 1. *For demand functions in $\mathcal{F}$ the following results hold: (a) WWA and GLD are equivalent; (b) WALD and RM are equivalent.*

Proof.
(a) WWA $\Rightarrow$ GLD. Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be in P, and take $\boldsymbol{f}(\boldsymbol{p},w)$ and $\boldsymbol{f}(\boldsymbol{q},w')$ where $w' = \boldsymbol{q}\cdot\boldsymbol{f}(\boldsymbol{p},w)$; clearly $\boldsymbol{f}(\boldsymbol{q},w') = \boldsymbol{s}(\boldsymbol{q})$, the compensated demand at point $(\boldsymbol{p},w)$. Since $\boldsymbol{q}\cdot\boldsymbol{f}(\boldsymbol{p},w) = w'$, WWA implies that $\boldsymbol{p}\cdot\boldsymbol{f}(\boldsymbol{q},w') \geq w$, i.e. $\boldsymbol{p}\cdot\left[\boldsymbol{s}(\boldsymbol{q}) - \boldsymbol{f}(\boldsymbol{p},w)\right] \geq 0$.[5]

GLD $\Rightarrow$ WWA. We will show that when WWA is violated then GLD is not satisfied. Let us assume that WWA is violated, i.e. there exists $(\boldsymbol{p},w)$ and $(\boldsymbol{q},w')$ such that $\boldsymbol{q}\cdot\boldsymbol{f}(\boldsymbol{p},w) \leq w'$ and $\boldsymbol{p}\cdot\boldsymbol{f}(\boldsymbol{q},w') < w$. Then, by Lemma 1 there exists $0 < t^* \leq 1$ such that $\boldsymbol{q}\cdot\boldsymbol{f}(t^*\boldsymbol{p},w) = w'$. Let us consider the effect on demand of a price change from $t^*\boldsymbol{p}$ to $\boldsymbol{q}$ and notice that $\boldsymbol{f}(\boldsymbol{q},w')$ is the compensated demand at point $(t^*\boldsymbol{p},w)$, indeed $\boldsymbol{s}(\boldsymbol{q}) = \boldsymbol{f}(\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{f}(t^*\boldsymbol{p},w)) = \boldsymbol{f}(\boldsymbol{q},w')$. We will show that GLD is violated, i.e.

$$t^*\boldsymbol{p}\cdot\left[\boldsymbol{s}(\boldsymbol{q}) - \boldsymbol{f}(t^*\boldsymbol{p},w)\right] < 0.$$

In fact the left-hand side is equal to $t^*\boldsymbol{p}\cdot\boldsymbol{f}(\boldsymbol{q},w') - w$, where t^* is not greater than 1 and, by assumption, $\boldsymbol{p}\cdot\boldsymbol{f}(\boldsymbol{q},w') < w$.

(b) That WALD $\Rightarrow$ RM is immediate from the definitions. We shall prove the converse.

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[5] This is similar to the proof of Lemma 1.(3) in Kihlstrom *et al.* (1976).

RM $\Rightarrow$ WALD. Let us assume that $\boldsymbol{q} \cdot \boldsymbol{f}(\boldsymbol{p}, w) \leq w$; we have to show that $\boldsymbol{p} \cdot \boldsymbol{f}(\boldsymbol{q}, w) \geq w$. By Lemma 1 there exists $0 < t^* \leq 1$ such that $\boldsymbol{q} \cdot \boldsymbol{f}(t^*\boldsymbol{p}, w) = w$ or equivalently $(\boldsymbol{q} - t^*\boldsymbol{p}) \cdot \boldsymbol{f}(t^*\boldsymbol{p}, w) = 0$. Then, from RM we have $(\boldsymbol{q} - t^*\boldsymbol{p}) \cdot \boldsymbol{f}(\boldsymbol{q}, w) \leq 0$, that is

$$t^*\boldsymbol{q} \cdot \boldsymbol{f}(\boldsymbol{p}, w) \geq w.$$

Then since $0 < t^* \leq 1$ it must be $\boldsymbol{q} \cdot f(\boldsymbol{p}, w) \geq w$.

∎

By noting that WWA and WALD are exactly the same thing when demand functions are homogeneous, an immediate corollary of Theorem 1 is:

COROLLARY 1. *For demand functions in $\mathcal{F}_h$ the properties WWA, GLD and RM are equivalent.*

Corollary 1 is the *finite* analog of the result of equivalence between WWA, NSD and ∂RM for continuously differentiable and homogeneous demand functions established partially by Kihlstrom *et al.* (1976) and by Hildenbrand and Jerison (1989), Th. 1. Finally, it can be noticed that Theorem 1 was proved without resorting to homogeneity of demand functions. In the next section we shall discuss the role played by homogeneity in the analysis of the weak axioms.

3. Some Remarks on Homogeneity

In a recent paper, John (1991) proved that WWA implies homogeneity.[6] Clearly if one can show that Wald's Weak Axiom or Restricted Monotonicity do not imply homogeneity of the demand function then the equivalence stated in Corollary 1 cannot be extended any further. Here is the example.

Example. Let us consider the function

$$\boldsymbol{f}(\boldsymbol{p}, w) = \Big(\frac{\log(w+1)}{p_1} \qquad \frac{w - \log(w+1)}{p_2}\Big),$$

for all $\boldsymbol{p}$ and w. It is easily seen that $\boldsymbol{f}(\boldsymbol{p}, w)$ is continuous, satisfies the budget constraint, but it is not homogeneous. The expression $(\boldsymbol{q} - \boldsymbol{p}) \cdot [\boldsymbol{f}(\boldsymbol{q}, w) - \boldsymbol{f}(\boldsymbol{p}, w)]$, for $\boldsymbol{q} \neq \boldsymbol{p}$ is equal to

$$-\log(w+1)\frac{(q_1 - p_1)^2}{p_1 q_1} - (w - \log(w+1))\frac{(q_2 - p_2)^2}{p_2 q_2} < 0,$$

6 The proof of this result is taken from John (1991) and is given in the appendix.

which is clearly negative since $w > \log(w+1)$ for $w > 0$. The demand function is then strictly monotone and certainly satisfies RM and WALD.[7]

This very simple example allows us to make some interesting remarks. In the first place, irrespective of whether demand functions are differentiable or not, even the 'Law of Demand' in its strongest version is not a sufficient condition for homogeneity. Therefore, if one is interested in modeling consumer behaviour but is not prepared to assume absence of money illusion one can still retain some degree of consistency of choice by adopting Wald's (Weak) Axiom or any other property related to the monotonicity of the demand function.

On the other hand, if one is interested in properties of demand functions related to the monotonicity of the Slutsky compensated demand, such as WWA or Samuelson's Weak Axiom of Revealed Preference, one is left less freedom in modeling consumer behaviour since he cannot help assuming implicitly homogeneity.

Furthermore, WALD and in general monotonicity properties seem to be more suitable assumptions for *market* demand functions, than WWA or WARP. Indeed, if relative income distribution is not fixed then homogeneity of market demand is not guaranteed by homogeneity of the individual demand functions.

4. Differential Characterization of the Weak Axioms

In this section we shall derive the differential characterization of the weak axioms by exploiting the equivalence with the notions of monotonicity established in section 2. The equivalence between WWA and NSD for homogeneous demand functions was originally proved by Kihlstrom, Mas-Colell and Sonnenschein (1976), and subsequently by Hildenbrand and Jerison (1989). The equivalence between WALD and ∂RM was shown by John (1991) and Hildenbrand (1993) by using the same argument as in Hildenbrand and Jerison (1989). In this section we shall prove both of these equivalence results by resorting to an alternative Ωmethod of proof. These results are essentially based on the following lemma.

[7] Notice also that the demand function is continuously differentiable and its Jacobian,

$$\partial_p f(p, w) = \begin{pmatrix} -\frac{\log(w+1)}{p_1^2} & 0 \\ 0 & -(\frac{w-\log(w+1)}{p_2^2}) \end{pmatrix},$$

is negative definite.

LEMMA 2. *Let $g: I \to \mathbb{R}$, with $I \subset \mathbb{R}$, be a differentiable function with $g(0) = g'(0) = 0$. If there exists $\epsilon > 0$ such that $g(t) > 0$ for all $t \in (0, \epsilon)$ then there exists $\hat{t} \in (0, \epsilon)$ such that*

$$g'(\hat{t}) \geq \frac{g(\hat{t})}{\hat{t}}. \tag{1}$$

Proof.
Let us assume that there exists an $\epsilon > 0$ such that $g(t) > 0$ for all $t \in (0, \epsilon)$ but there is no $\hat{t} \in (0, \epsilon)$ such that (1) holds; then, it must be

$$g'(t) < \frac{g(t)}{t}, \qquad \text{for all} \qquad t \in (0, \epsilon). \tag{2}$$

Let us consider the function $g(t)/t$ defined on the closed interval $[0, \epsilon]$ and set it equal to zero for $t = 0$. Then, $g(t)/t$ is continuous on $[0, \epsilon]$ since $g(t)$ is continuous and, by assumption, $\lim_{t \to 0} g(t)/t = g'(0) = 0$; moreover, the function is strictly positive for $t \in (0, \epsilon)$. This function has a maximum and a minimum in the closed interval $[0, \epsilon]$; clearly, the minimum is 0 and the maximum is strictly positive and is denoted by $\bar{g}$. Let us take $\delta \in (0, \bar{g})$ and consider the set $B = \{t \in [0, \epsilon] \mid g(t)/t \geq \delta\}$. By continuity, the set B is not empty and closed, and $t_0 = \min B$ is well defined. Moreover,[8] $0 < t_0 < \epsilon$, $g(t_0)/t_0 = \delta$ and

$$\frac{g(t)}{t} < \delta \qquad \text{for all} \qquad t \in (0, t_0). \tag{3}$$

Then (2) and (3) yields

$$g'(t) < \delta \qquad \text{for all} \qquad t \in (0, t_0). \tag{4}$$

But this cannot be true. Indeed, by the mean value theorem we have $g(t_0) = g'(\tilde{t})t_0$ for some $\tilde{t} \in (0, t_0)$, that is

$$g'(\tilde{t}) = \frac{g(t_0)}{t_0} = \delta$$

so that (4) is violated. This contradiction completes the proof.

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Now we are ready to give the first result of this section.

[8] That, $t_0 \neq 0$ is trivial; that $t_0 \neq \epsilon$ is obvious from $\delta < \bar{g}$ and a continuity argument.

THEOREM 2. *For continuously differentiable demand functions RM and ∂RM are equivalent.*

Proof.[9]
RM $\Rightarrow$ ∂RM. Let us set $\boldsymbol{v} = \boldsymbol{q} - \boldsymbol{p}$ and let $\boldsymbol{v} \cdot \boldsymbol{f}(\boldsymbol{p}) = 0$. Then define $\boldsymbol{p}(t) = \boldsymbol{p} + t\boldsymbol{v}$ and the function

$$g(t) = \boldsymbol{v} \cdot \boldsymbol{f}(\boldsymbol{p}(t)).$$

Since $(\boldsymbol{p}(t) - \boldsymbol{p}) = t\boldsymbol{v}$ we have $(\boldsymbol{p}(t) - \boldsymbol{p}) \cdot \boldsymbol{f}(\boldsymbol{p}) = 0$ for all $t > 0$ and by RM

$$(\boldsymbol{p}(t) - \boldsymbol{p}) \cdot \boldsymbol{f}(\boldsymbol{p}(t)) = tg(t) \leq 0$$

for all $t > 0$. Then, $g(0) = 0$ and $g(t) \leq 0$ for all $t > 0$ imply that

$$g'(0) = \boldsymbol{v} \cdot \partial_p \boldsymbol{f}(\boldsymbol{p})\, \boldsymbol{v} \leq 0.$$

∂RM $\Rightarrow$ RM.
Let us assume that ∂RM holds but RM is violated; then, without loss of generality we can assume that,[10] for some $\boldsymbol{v}$ and $\boldsymbol{p}$, we have $\boldsymbol{v} \cdot \boldsymbol{f}(\boldsymbol{p}) = \boldsymbol{v} \cdot \partial \boldsymbol{f}(\boldsymbol{p})\boldsymbol{v} = 0$ and there exists $t_0 > 0$ such that

$$g(t) = \boldsymbol{v} \cdot \boldsymbol{f}(\boldsymbol{p} + t\boldsymbol{v}) > 0, \qquad \text{for all} \qquad t \in (0, t_0) \tag{5}$$

Let us consider the function

$$h(t) = \frac{\boldsymbol{p} \cdot \partial_p \boldsymbol{f}(\boldsymbol{p}_t)\, \boldsymbol{p}_t}{w};$$

since $\boldsymbol{f}(\boldsymbol{p})$ is continuously differentiable, the function $h(t)$ is continuous; by budget identity, $\boldsymbol{p} \cdot \partial \boldsymbol{f}(\boldsymbol{p}) = -\boldsymbol{f}(\boldsymbol{p})$ so that $h(0) = -1$. Continuity of $h(t)$ means that there exists $\eta > 0$ such that $|h(t) - h(0)| < 1$ or

$$-2 < h(t) < 0, \qquad \text{for all} \qquad t \in (-\eta, \eta). \tag{6}$$

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[9] For notational convenience we have dropped income as an argument of the demand function since income is fixed.

[10] If RM is violated, then there exist prices $\boldsymbol{p}'$ and $\boldsymbol{q}$ such that $(\boldsymbol{q} - \boldsymbol{p}') \cdot \boldsymbol{f}(\boldsymbol{p}') = 0$ and $(\boldsymbol{q} - \boldsymbol{p}') \cdot \boldsymbol{f}(\boldsymbol{q}) > 0$. Let us set $\boldsymbol{v} = \boldsymbol{q} - \boldsymbol{p}'$, $\boldsymbol{p}'_t = \boldsymbol{p}' + t\boldsymbol{v}$ an define the function $k(t) = \boldsymbol{v} \cdot \boldsymbol{f}(\boldsymbol{p}'_t)$. By assumption, $k(0) = 0$ and $k(1) > 0$, so that by continuity there exists $\bar{t} \in [0, 1)$ such that $k(\bar{t}) = 0$ and $k(t) > 0$ for all $t \in (\bar{t}, 1]$. Let us set $t_0 = 1 - \bar{t}$, $\boldsymbol{p} = \boldsymbol{p}' + \bar{t}\boldsymbol{v}$ and define

$$g(\tau) = \boldsymbol{v} \cdot \boldsymbol{f}(\boldsymbol{p} + \tau\boldsymbol{v}) = k(\bar{t} + \tau)$$

Clearly, $g(0) = \boldsymbol{v} \cdot \boldsymbol{f}(\boldsymbol{p}) = 0$, and $g(\tau) > 0$ for all $\tau \in (0, t_0]$. Moreover, by ∂RM, $g(0) = 0$ implies $g'(0) = \boldsymbol{v} \cdot \partial \boldsymbol{f}(\boldsymbol{p})\boldsymbol{v} \leq 0$, but since $g(\tau) > 0$ for $\tau > 0$ it must be $g'(0) = 0$ as stated above.

Therefore, let us define $\epsilon = \min\{\eta, t_0\}$ and notice that for any $t \in (0, \epsilon)$ the vector $\boldsymbol{v}$ can be expressed as

$$\boldsymbol{v} = \boldsymbol{u}(t) + \alpha(t)\boldsymbol{p}(t), \tag{7}$$

where $\boldsymbol{u}(t)$ is orthogonal to $\boldsymbol{f}(\boldsymbol{p}(t))$, i.e. $\boldsymbol{u}(t) \cdot \boldsymbol{f}(\boldsymbol{p}(t)) = 0$, and $\alpha(t) = \boldsymbol{v} \cdot \boldsymbol{f}(\boldsymbol{p}(t))/w = g(t)/w$. Consider then the expression

$$\boldsymbol{u}(t) \cdot \partial_p \boldsymbol{f}(\boldsymbol{p}(t))\, \boldsymbol{u}(t) = [\boldsymbol{v} - \alpha(t)\boldsymbol{p}(t)] \cdot \partial_p \boldsymbol{f}(\boldsymbol{p}(t))\, \boldsymbol{u}(t)$$

where we have substituted $\boldsymbol{u}(t)$ from (7). By budget identity and by orthogonality of $\boldsymbol{u}(t)$ we obtain

$$\boldsymbol{u}(t) \cdot \partial_p \boldsymbol{f}(\boldsymbol{p}(t))\, \boldsymbol{u}(t) = \boldsymbol{v} \cdot \partial_p \boldsymbol{f}(\boldsymbol{p}(t))\, \boldsymbol{v} - \alpha(t)\boldsymbol{v} \cdot \partial_p \boldsymbol{f}(\boldsymbol{p}(t))\, \boldsymbol{p}(t).$$

Noting that the first term of RHS is equal to $g'(t)$ and using the definition of $\alpha(t)$ yields

$$\boldsymbol{u}(t) \cdot \partial_p \boldsymbol{f}(\boldsymbol{p}(t))\, \boldsymbol{u}(t) = g'(t) - \frac{g(t)}{w} \boldsymbol{v}\, \partial_p \boldsymbol{f}(\boldsymbol{p}(t))\, \boldsymbol{p}(t).$$

Since $g(t) > 0$, the sign of the above expression does not change by multiplying it by $t/g(t)$, and we can focus on the sign of

$$\frac{g'(t)}{g(t)/t} - \frac{t\boldsymbol{v} \cdot \partial_p \boldsymbol{f}(\boldsymbol{p}(t))\, \boldsymbol{p}(t)}{w} \tag{8}$$

By budget identity and by the definition of $\boldsymbol{p}(t)$ we obtain

$$-t\boldsymbol{v}\partial_p \boldsymbol{f}(\boldsymbol{p}(t)) = \boldsymbol{p}\, \partial_p \boldsymbol{f}(\boldsymbol{p}(t)) + \boldsymbol{f}(\boldsymbol{p}(t));$$

substituting the above expression in (8) yields

$$\frac{g'(t)}{g(t)/t} + 1 + \frac{\boldsymbol{p} \cdot \partial_p \boldsymbol{f}(\boldsymbol{p}(t))\, \boldsymbol{p}(t)}{w}.$$

By Lemma 2 there exists $\hat{t} \in (0, \epsilon)$ such that the first term is greater equal than 1, so that for $t = \hat{t}$ the above expression is greater equal than

$$2 + h(\hat{t}) \tag{9}$$

where we applied the definition of $h(t)$; then, by (6) and $\hat{t} < \eta$, it follows that (9) is strictly positive. Therefore, we have established that $u(\hat{t}) \cdot \boldsymbol{f}(\boldsymbol{p}(\hat{t})) = 0$, but

$$\boldsymbol{u}(\hat{t}) \cdot \partial_p \boldsymbol{f}(\boldsymbol{p}(\hat{t}))\, \boldsymbol{u}(\hat{t}) > 0$$

which contradicts ∂RM.

■

From Theorem 1.(b) and Theorem 2 we obtain immediately the equivalence between WALD and ∂RM. This is an interesting result since $\boldsymbol{f}(\boldsymbol{p}, w)$ can be interpreted as a *market* demand and, as shown by Hildenbrand (1993), WALD for market demand guarantees uniqueness of equilibrium in regular economies.

Let us turn to the second result of this section, the equivalence between GLD and NSD. It is clear that this result can be derived by Theorems 1 and 2, by $WWA \Rightarrow$ homogeneity and by showing the equivalence between NSD and ∂RM plus homogeneity[11]. However, we shall follow a direct route which is essentially based on the same method of proof of Theorem 2.

THEOREM 3. *For continuously differentiable demand functions GLD and NSD are equivalent.*

Proof.
GLD $\Rightarrow$ NSD. Let us set $\boldsymbol{v} = \boldsymbol{q} - \boldsymbol{p}$, $\boldsymbol{p}(t) = \boldsymbol{p} + t\boldsymbol{v}$ and define the function

$$g(t) = \boldsymbol{v} \cdot [\boldsymbol{s}(\boldsymbol{p}(t)) - \boldsymbol{f}(\boldsymbol{p}, w)]$$

where $\boldsymbol{s}(\boldsymbol{p}(t))$ is the Slutsky compensated demand at point $(\boldsymbol{p}, w)$. Then, $g(0) = 0$ by definition and by GLD

$$t\, g(t) = (\boldsymbol{p}(t) - \boldsymbol{p}) \cdot [\boldsymbol{s}(\boldsymbol{p}(t)) - \boldsymbol{f}(\boldsymbol{p}, w)] \leq 0$$

for $t > 0$. Then, $g(0) = 0$ and $g(t) \leq 0$ for $t > 0$ imply that

$$g'(0) = \boldsymbol{v} \cdot S(\boldsymbol{p}, w)\boldsymbol{v} \leq 0.$$

NSD $\Rightarrow$ GLD. Let us consider first the derivative of $g(t)$, i.e.

$$g'(t) = \boldsymbol{v} \cdot [\partial_p \boldsymbol{f}(\boldsymbol{p}(t), w(t)) + \partial_w \boldsymbol{f}(\boldsymbol{p}(t), w(t)) \boldsymbol{f}(\boldsymbol{p}, w)^T]\ \boldsymbol{v}$$

[11] The last result is already available: ∂RM plus homogeneity $\Rightarrow$ NSD is the step (ii) - (iii) in Th.1 of Hildenbrand and Jerison (1989); NSD $\Rightarrow$ ∂RM is trivial and NSD $\Rightarrow$ homogeneity was proved by John (1991) and Hildenbrand (1993).

where $w(t) = \boldsymbol{p}(t) \cdot \boldsymbol{f}(\boldsymbol{p}, w)$. By adding and subtracting

$$\boldsymbol{v} \cdot [\partial_w \boldsymbol{f}(\boldsymbol{p}(t), w(t)) \boldsymbol{f}(\boldsymbol{p}(t), w(t))^T] \, \boldsymbol{v}$$

in the RHS of the above expression we obtain

$$g'(t) = \boldsymbol{v} \cdot S(\boldsymbol{p}(t), w(t))\boldsymbol{v} - g(t)\boldsymbol{v} \cdot \partial_w \boldsymbol{f}(\boldsymbol{p}(t), w(t)), \tag{10}$$

where $S(\boldsymbol{p}(t), w(t))$ is the Slutsky matrix at point $(\boldsymbol{p}(t), w(t))$.

Let us assume that NSD holds but GLD is violated; then, without loss of generality, there exist $\boldsymbol{p}$, w and $\boldsymbol{v}$ such that $g(0) = 0$ and there exists $t_0 > 0$ such that $g(t) > 0$ for all $0 < t \leq t_0$.[12]

Consider now the function

$$h(t) = \boldsymbol{p} \cdot \partial_w \boldsymbol{f}(\boldsymbol{p}(t), w(t)).$$

[12] If GLD is violated, then there exist prices $\boldsymbol{p}_0$, $\boldsymbol{q}$ and income w_0 such that $(\boldsymbol{q} - \boldsymbol{p}_0) \cdot [\boldsymbol{s}(\boldsymbol{q}; \boldsymbol{p}_0, w_0) - \boldsymbol{f}(\boldsymbol{p}_0, w_0)] > 0$, where we made explicit that the Slutsky compensated demand is referred to point $(\boldsymbol{p}_0, w_0)$. Now, set $\boldsymbol{v}_0 = \boldsymbol{q} - \boldsymbol{p}_0$, $\boldsymbol{p}_0(t) = \boldsymbol{p}_0 + t\boldsymbol{v}_0$ and define

$$k(t) = \boldsymbol{v}_0 \cdot [\boldsymbol{s}(\boldsymbol{p}(t); \boldsymbol{p}_0, w_0) - \boldsymbol{f}(\boldsymbol{p}_0, w_0)];$$

clearly, $k(0) = 0$ and $k(1) > 0$, and by continuity there exists $\bar{t} \in [0,1)$ such that $k(\bar{t}) = 0$ and $k(t) > 0$ for $t \in (\bar{t}, 1]$. Now notice that $k(\bar{t}) = 0$ means

$$\boldsymbol{v}_0 \cdot \boldsymbol{f}(\bar{\boldsymbol{p}}_0; \bar{w}_0) = \boldsymbol{v}_0 \cdot \boldsymbol{f}(\boldsymbol{p}_0, w_0) \tag{*}$$

where we have set $\bar{\boldsymbol{p}}_0 = \boldsymbol{p}_0 + \bar{t}\boldsymbol{v}_0$ and $\bar{w}_0 = \bar{\boldsymbol{p}}_0 \cdot \boldsymbol{f}(\boldsymbol{p}_0, w_0)$. Then by budget identity and (*) we have $\boldsymbol{p}_0 \cdot \boldsymbol{f}(\bar{\boldsymbol{p}}_0; \bar{w}_0) = \boldsymbol{p}_0 \cdot \boldsymbol{f}(\boldsymbol{p}_0, w_0)$ therefore

$$\boldsymbol{p}_0(t) \cdot \boldsymbol{f}(\bar{\boldsymbol{p}}_0; \bar{w}_0) = \boldsymbol{p}_0(t) \cdot \boldsymbol{f}(\boldsymbol{p}_0, w_0) \tag{**}$$

for all t. Now set $\boldsymbol{p} = \bar{\boldsymbol{p}}_0$, $w = \bar{w}_0$ and consider prices $\boldsymbol{p}$, $\boldsymbol{q}$ and income w. Set $t_0 = 1 - \bar{t}$, $v = \boldsymbol{q} - \boldsymbol{p} = (1 - \bar{t})\boldsymbol{v}_0$ and $\boldsymbol{p}(\tau) = \boldsymbol{p} + \tau\boldsymbol{v}$ and consider the function

$$g(\tau) = \boldsymbol{v} \cdot [\boldsymbol{s}(\boldsymbol{p}(\tau); \boldsymbol{p}, w) - \boldsymbol{f}(\boldsymbol{p}, w)] \tag{***}$$

with $0 \leq \tau \leq t_0$. Clearly, by equation (**), $\boldsymbol{s}(\boldsymbol{p}(\tau); \boldsymbol{p}, w) = \boldsymbol{s}(\boldsymbol{p}(\tau); \boldsymbol{p}_0, w_0)$; adding and subtracting $\boldsymbol{f}(\boldsymbol{p}_0, w_0)$ within the square brackets of (***) yields

$$g(\tau) = (1 - \bar{t})k(\bar{t} + \tau) + (1 - \bar{t})k(\bar{t}) = (1 - \bar{t})k(\bar{t} + \tau),$$

which means $g(0) = 0$ and $g(\tau) > 0$ for all $\tau \in (0, t_0]$; notice also that, by (10), NSD implies $g'(0) \leq 0$.

Since the demand function is continuously differentiable the function $h(t)$ is continuous; moreover, by budget identity $h(0) = 1$. Then by continuity there exists $\eta > 0$ such that $|h(t) - h(0)| < 1$, or equivalently

$$0 < h(t) < 2 \tag{11}$$

for all $t \in (-\eta, \eta)$. Let us define $\epsilon = \min\{\eta, t_0\}$ and rearrange (10) so as to yield

$$\boldsymbol{v} \cdot S(\boldsymbol{p}(t), w(t))\boldsymbol{v} = g'(t) + g(t)\boldsymbol{v} \cdot \partial_w \boldsymbol{f}(\boldsymbol{p}(t), w(t)). \tag{12}$$

We shall show that there exists $\bar{t} \in (0, \epsilon)$ such that NSD is violated. Indeed, since $g(t)$ is strictly positive in the relevant interval, the sign of $\boldsymbol{v} \cdot S(\boldsymbol{p}(t), w(t))\boldsymbol{v}$ is the same as that of the expression that we obtain by multiplying the RHS of (12) by $t/g(t)$, i.e.

$$\frac{g'(t)}{g(t)/t} + t\boldsymbol{v} \cdot \partial_w \boldsymbol{f}(\boldsymbol{p}(t), w(t)). \tag{13}$$

Then, by budget identity and definition of $\boldsymbol{p}(t)$ the second term of (13) is equal to $1 - h(t)$; moreover, by Lemma 2 there exists $\bar{t} \in (0, \epsilon)$ such that the first term is greater equal than 1. It follows that for $t = \bar{t}$, (13) is greater equal than

$$2 - h(\bar{t})$$

which in turns, by (11) is strictly positive. Therefore, we have shown that the Slutsky matrix, $S(\boldsymbol{p}(\bar{t}), w(\bar{t}))$, is not negative semi definite and this contradiction establishes the result.

■

From Theorem 1.(a) and Theorem 3 we obtain immediately the equivalence between WWA and NSD. It can be noticed that, as in the proofs of theorems 1 and 2, homogeneity of demand functions has not been assumed. However, from WWA $\Rightarrow$ homogeneity we obtain immediately NSD $\Rightarrow$ homogeneity.

5. Summary and Conclusions

The content of Theorem 1, 2 and 3 can be summarized as follows:

$$WALD \iff RM \iff \partial\text{RM}$$

and

$$WWA \iff GLD \iff NSD$$

Furthermore, in section 3, we have established that WWA and WALD are equivalent *if and only if* demand functions are homogeneous. It is then immediate to derive the equivalence between WWA, NSD and ∂RM for continuously differentiable homogeneous demand functions (Th. 1 in Hildenbrand and Jerison (1989)). In addition, we have that this equivalence result cannot be strengthened, that is, it cannot hold for non homogeneous functions.

Furthermore, from WWA $\Rightarrow$ homogeneity we immediately obtain NSD $\Rightarrow$ homogeneity; also, it is immediate to see that ∂RM plus homogeneity is equivalent to NSD (these results are also proved by John (1991) and Hildenbrand (1994)).

Appendix

Proof of: WWA implies homogeneity.[13]

We have to prove that $\boldsymbol{f}(\boldsymbol{p},w)=\boldsymbol{f}(\alpha\boldsymbol{p},\alpha w)$, for $\alpha>0$. It is sufficient to show that

$$\boldsymbol{q}\cdot\left[\boldsymbol{f}(\boldsymbol{p},w)-\boldsymbol{f}(\alpha\boldsymbol{p},\alpha w)\right]\leq 0,$$

for all $\boldsymbol{q}\in P$. Let us normalize $\boldsymbol{q}$ so that $\boldsymbol{q}\cdot\boldsymbol{f}(\alpha\boldsymbol{p},\alpha w)=w$; therefore, to prove the proposition one has to show that $\boldsymbol{q}\cdot\boldsymbol{f}(\boldsymbol{p},w)\leq w$. Consider the price vectors $\boldsymbol{q}(t)=t\boldsymbol{q}+(1-t)\boldsymbol{p}$, with $t\in[0,1]$. Clearly, $\boldsymbol{q}(t)\cdot\boldsymbol{f}(\alpha\boldsymbol{p},\alpha w)=w$ so that both WWA and GLD imply

$$\boldsymbol{p}\cdot\boldsymbol{f}(\boldsymbol{q}(t),w)\geq w. \tag{1}$$

From budget identity and $\boldsymbol{q}(t)\cdot\boldsymbol{f}(\boldsymbol{q}(t),w)=w$ one gets

$$t\big[\boldsymbol{q}\cdot\boldsymbol{f}(\boldsymbol{q}(t),w)-w\big]+(1-t)\big[\boldsymbol{p}\cdot\boldsymbol{f}(\boldsymbol{q}(t),w)-w\big]=0;$$

the above expression and inequality (1) imply that $\boldsymbol{q}\cdot\boldsymbol{f}(\boldsymbol{q}(t),w)\leq w$, then, letting t go to 0, by continuity $\boldsymbol{q}\cdot\boldsymbol{f}(\boldsymbol{p},w)\leq w$.

■

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[13] The proof is taken from John (1991).

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