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Demand and generalized monotonicity

by

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Abstract: This paper reviews some important results on the characterization of mild versions of the Weak Axiom of Revealed Preference by using a notion of generalized monotonicity proposed in the recent mathematical literature. The results are derived in a unified framework which allows for direct economic interpretations and relatively simple and short proofs.

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DEMAND AND GENERALIZED MONOTONICITY

Introduction

This paper reviews some important results of the theory of demand based on revealed preference axioms. The focus is on two weaker versions of Samuelson's Weak Axiom of Revealed Preferences (WARP) called respectively the Weak Axiom and Wald's Axiom. These axioms are useful tools for modeling consumer behaviour in many circumstances. They are milder rationality requirements than the hypothesis of preference maximization and still impose similar restrictions on consumer demand responses to price changes. These axioms also provide powerful properties for dealing with aggregate and market demand functions. In fact, for many types of economies, the convexity of the equilibrium set follows from these properties, therefore, under the hypothesis of regularity, the weak axioms are strong enough to ensure uniqueness of equilibrium.

The present paper derives various results about the characterization of the weak axioms. Though many of the following results are already available in the literature, this paper proposes a unified perspective which allows for direct economic interpretations and relatively simple and short proofs.

Both the Weak Axiom and Wald's Axiom impose on demand functions a property of generalized monotonicity known in the mathematical literature as pseudomonotonicity. In section 2, by using this formal property, it is shown that the weak axioms are equivalent to 'compensated' versions of the Law of Demand. In section 3, for a particular class of functions including demand functions, we characterize pseudomonotonicity in terms of a property of the Jacobian matrix. In section 4, the previous result is used to derive differential characterizations of both the Weak Axiom and Wald's Axiom. The connections to the existing literature are discussed in the last part of this section. Finally, in section 5, we illustrate the relevance of the above results for market demand theory.

1. Preliminary definitions

Demand is defined as a continuous function $f: \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \mathbb{R}^n_{++}$ satisfying budget identity, i.e. $p \cdot f(p, w) = w$, where p is a vector of prices and w is income or expenditure. We say that demand is homogeneous if f is homogeneous of degree zero in p and w, i.e. $f(\alpha p, \alpha w) = f(p, w)$ for any $\alpha > 0$.

Definition 1.
A demand function satisfies the Weak Axiom (WA) if

$$q \cdot f(p, w) \leq w' \qquad \text{implies} \qquad p \cdot f(q, w') \geq w. \tag{1}$$

The Weak Axiom is a condition on demand functions similar to the Weak Axiom of Revealed Preference (WARP) originally proposed by Samuelson (1938). WARP requires that the 'revealed preference relation' is antisymmetric, in the sense that, whenever

$$q \cdot f(p, w) \leq w' \qquad \text{and} \qquad p \cdot f(q, w') \leq w$$

it follows that $f(p, w) = f(q, w')$. On the other hand, the Weak Axiom only requires that $q \cdot f(p, w) = w'$ and $p \cdot f(q, w') = w$. Clearly, a demand function satisfying WARP also satisfies WA, but, as it was shown by Kihlstrom, Mas-Colell and Sonnenschein (1976), the converse is not true. Thus, WA is actually a weaker condition than WARP.

A still milder requirement on demand functions is Wald's Axiom, which is the fixed income version of the Weak Axiom.

Definition 2.
A demand function satisfies Wald's Axiom (WD) if

$$q \cdot f(p, w) \leq w \qquad \text{implies} \qquad p \cdot f(q, w) \geq w. \tag{2}$$

When demand is homogeneous, Wald's Axiom and the Weak Axiom are equivalent conditions. Indeed, let f be an homogeneous demand function satisfying Wald's Axiom; if $q \cdot f(p, w) \leq w'$, then $(w/w')q \cdot f(p, w) \leq w$, hence by WD we have $p \cdot f((w/w')q, w) \geq w$ and finally by homogeneity $p \cdot f(q, w') \geq w$. Thus demand satisfies the Weak Axiom.

The economic content of the two axioms is not immediately evident from their definitions; however, as we shall see below, there is a more intuitive interpretation of these axioms as 'compensated' versions of the 'Law of Demand'.

In a single market the Law of Demand corresponds to the case of a downward sloping demand curve, that is, the case where quantities of the good vary inversely with its price. With multiple markets, the Law of Demand is defined as the case where the vector of demand changes and the vector of price changes move in opposite directions. More formally, the Law of Demand asserts that, for any w, the function $p \mapsto f(p, w)$ is monotone, i.e.

$$(q - p) \cdot [f(q, w) - f(p, w)] \leq 0$$

Let us introduce the following definitions.

Definition 3.
A demand function satisfies the *Compensated Law of Demand* (CLD), if

$$(q - p) \cdot [f(q, q \cdot f(p, w)) - f(p, w)] \leq 0 \tag{3}$$

for all p, q and w.

Definition 4.
A demand function satisfies the *Price Compensated Law of Demand* (PCLD), if

$$(q - p) \cdot [f(q, w) - f(p, w)] \leq 0 \tag{4}$$

for all p, q and w such that $(q - p) \cdot f(p, w) = 0$

The Compensated Law of Demand states that the vector of changes in compensated demand and the vector of price changes move in opposite directions; in particular it implies that, once compensated in terms of income, a consumer cannot respond to an increase of the price of one good by rising its demand.

The Price Compensated Law of Demand has a similar interpretation, but only for price changes which do not require any income compensation, that is only price changes that keep constant the expenditure for the initially demanded bundle of goods.

In the following section we derive a first characterization of the weak axioms in terms of these 'compensated' versions of the Law of Demand and in terms of a notion of generalized monotonicity.

2. A characterization of the weak axioms

Although the weak axioms do not imply the Law of Demand they still impose some requirements of monotonicity on demand functions. Among the various notions of 'generalized monotonicity' analyzed in the recent mathematical literature,[1] the one best suited for our purpose is the property of pseudomonotonicity.

DEFINITION 5.
A function $F\colon D \to \mathbb{R}^n$, where D is a subset of $\mathbb{R}^n$, is *pseudomonotone* if

$$(y-x)\cdot F(x) \le 0 \qquad \text{implies} \qquad (y-x)\cdot F(y) \le 0 \tag{5}$$

Obviously, a monotone function is also pseudomonotone, but not vice versa. The interpretation of this property is particularly simple in the case of one variable functions, i.e. for $n = 1$. Indeed, although pseudomonotone functions are not necessarily downward sloping they have a 'sign preserving' property in the sense that they cannot change their sign more than once. From a graphical point of view, whenever a pseudomonotone function attains zero, its graph remains below the horizontal axis thereafter.[2] An example of a pseudomonotone function is depicted in Figure 1. This figure also shows that pseudomonotonicity is not preserved under affine transformations.

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[1] See, for example, Karamardian and Schiable (1990).

[2] On geometrical properties of generalized monotonicity notions in the one dimensional case see Karamardian, Schaible and Crouzeix (1993).

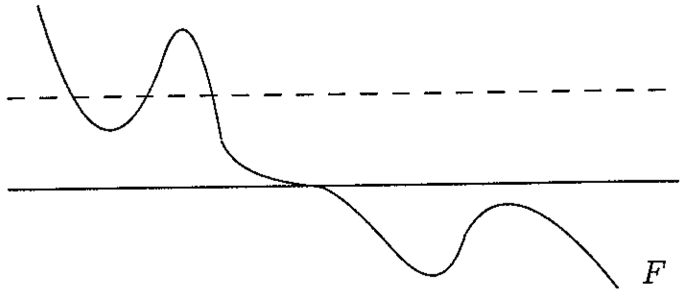

Fig. 1. A pseudomonotone function

The next result provides an equivalent definition of pseudomonotonicity for continuous maps with convex domain and will be used later on in the paper.

LEMMA 1.
Let $F\colon D \to \mathbb{R}^n$ be a continuous function and D a convex subset of $\mathbb{R}^n$. F is pseudomonotone if and only if

$$(y-x)\cdot F(x) = 0 \qquad \text{implies} \qquad (y-x)\cdot F(y) \le 0 \tag{6}$$

Proof. That (5) implies (6) is immediate. To prove the converse let us assume that

$$(y-x)\cdot F(x) < 0 \qquad \text{but} \qquad (y-x)\cdot F(y) > 0.$$

Let us set $v = y - x$ and consider the one variable function

$$g(t) = v \cdot F(x + tv)$$

for $t \in [0,1]$. Since $g(0) < 0$ and $g(1) > 0$, by continuity of g, there exists a $t' \in (0,1)$ such that $g(t') = 0$. Hence

$$(y - (x + t'v)) \cdot F(x + t'v) = (1 - t')g(t') = 0$$

and

$$(y - (x + t'v)) \cdot F(y) = (1 - t')g(1) > 0$$

and (6) is violated. Thus, (6) implies (5).

Q.E.D.

Having introduced the notion of pseudomonotonicity we are now ready to show its relationship with Wald's Axiom and the Price Compensated Law of Demand. By using budget identity, it is strightforward to see that Wald's Axiom amounts to requiring that, for any w, the function $p \mapsto f(p, w)$ is pseudomonotone; indeed, (2) can be written as

$$(q - p) \cdot f(p, w) \leq 0 \qquad \text{implies} \qquad (q - p) \cdot f(q, w) \leq 0 \quad .$$

On the other hand, by Lemma 1, we notice that pseudomonotonicity of $p \mapsto f(p, w)$ is equivalent to the Price Compensated Law of Demand. We bring together these results characterizing Wald's Axiom in the following proposition.

PROPOSITION 1.
Let f be a demand function. The following statements are equivalent:
a) f satisfies Wald's Axiom
b) f satisfies the Price Compensated Law of Demand
c) for any w, the map $p \mapsto f(p, w)$ is pseudomonotone.

Let us now turn to the analysis of the Weak Axiom and its relationships with the other properties. Clearly, the Weak Axiom imposes the requirement of pseudomonotonicity on demand functions since it is a stronger property than Wald's Axiom. However, as we shall see, it also places an additional restriction which is homogeneity. Before giving a complete characterization of the Weak Axiom let us first prove a preliminary result concerning the Compensated Law of Demand.

By using budget identity it is easy to see that the Compensated Law of Demand is equivalent to the following condition

$$p \cdot f(q, q \cdot f(p, w)) \geq w \qquad \text{for all } p,\ q \text{ and } w \tag{7}$$

Next we use this equivalent definition to show that CLD implies homogeneity of the demand function.

LEMMA 2.
Let f be a demand function satisfying the Compensated Law of Demand. Then f is homogeneous, i.e. $f(\alpha p, \alpha w) = f(p, w)$ for any $\alpha > 0$.

Proof. For any vector v in $\mathbb{R}^n$ let us consider the prices $q = p + tv$ for $t > 0$ and sufficiently close to zero. By using budget identity and the definition of q we obtain

$$q \cdot f(\alpha p, \alpha w) = w + tv \cdot f(\alpha p, \alpha w)$$

and

$$q \cdot f(q, q \cdot f(\alpha p, \alpha w)) = p \cdot f(q, q \cdot f(\alpha p, \alpha w)) + tv \cdot f(q, q \cdot f(\alpha p, \alpha w))$$

Equating the above identities and rearranging yields

$$tv \cdot [f(\alpha p, \alpha w) - f(q, q \cdot f(\alpha p, \alpha w))] = p \cdot f(q, q \cdot f(\alpha p, \alpha w)) - w \tag{$*$}$$

Since f satisfies CLD, from (7) and $(*)$ we obtain

$$v \cdot [f(\alpha p, \alpha w) - f(p + tv, w + tv \cdot f(\alpha p, \alpha w))] \geq 0$$

for any $t > 0$. By continuity of f we then have

$$v \cdot [f(\alpha p, \alpha w) - f(p, w)] \geq 0$$

and since this is true for any $v \in \mathbb{R}^n$ the vector in square brakets must be zero.

Q.E.D.

The following result is a characterization of the Weak Axiom.

PROPOSITION 2.
Let f be a demand function. The following statements are equivalent:
a) f satisfies the Weak Axiom
b) f satisfies the Compensated Law of Demand

c) f is homogeneous and the map $p \mapsto f(p,w)$ is pseudomonotone.

Proof. a) $\Rightarrow$ b). For any p, q and w set $w' = q \cdot f(p,w)$. By (1) we have $p \cdot f(q,w') \geq w$ and substituting w' yields (7). Hence, WA implies CLD.

b) $\Rightarrow$ c). Homogeneity of f follows from Lemma 2. Next, let $(q-p) \cdot f(p,w) = 0$ then by budget identity

$$q \cdot f(p,w) = w \tag{$*$}$$

hence (7) and $(*)$ yield

$$p \cdot f(q, q \cdot f(p,w)) = p \cdot f(q,w) \geq w$$

or $(q-p) \cdot f(q,w) \leq 0$; then by Lemma 1 the map $p \mapsto f(p,w)$ is pseudomonotone.

c) $\Rightarrow$ a). By Proposition 1 f satisfies Wald's Axiom. Hence, from homogeneity of f the result follows trivially.

Q.E.D.

Propositions 1 and 2 provide direct economic interpretations of the weak axioms in terms of 'compensated' versions of the Law of Demand. Based on these interpretations, the weak axioms appear to be appealing alternatives to the assumption of maximization of preferences under budget constraint in modeling consumer behaviour. Indeed, while the weak axioms are milder rationality requirements they still impose similar restrictions on demand functions, that is, price changes must have a non positive compensated effect on demand.

Propositions 1 and 2 also clarify the difference between the two axioms by spelling out in detail the formal properties they impose on demand. While pseudomonotonicity is a common property, homogeneity is imposed only by the Weak axiom and not by Wald's Axiom. To see this it is sufficient to consider the following example. The demand function $f\colon \mathbb{R}^3_{++} \to \mathbb{R}^2$ defined by

$$f(p,w) = \begin{pmatrix} \dfrac{\log(w+1)}{p_1} & \dfrac{w - \log(w+1)}{p_2} \end{pmatrix}$$

is not homogeneous, but is pseudomonotone (actually, it is strictly monotone), therefore, by Proposition 1, it satisfies Wald's Axiom.

The above results may be of some help in choosing appropriate assumptions to model consumer behaviour. There are many situations of interest in both theoretical and applied work where homogeneity of individual demand is not a justifiable assumption. This is the case, for example, when the analysis concerns a subgroup of goods and individual demand is modeled as a function of expenditure and prices of these goods only. A similar case arises when, in an intertemporal decision setting, individual demand behaviour in the current period is modeled as a function of current prices and income. In all the above situations individual demand is not homogeneous, therefore it cannot be expected to satisfy the Weak Axiom. On the contrary, Wald's Axiom is still an available option to model consumer behaviour.

3. A Differential characterization of pseudomonotone functions

As we have seen, a common feature of demand functions satisfying either the Weak Axiom or Wald's Axiom is pseudomonotonicity. In this section we concentrate on this property and derive an important result that will be used in the sequel. Pseudomonotone differentiable functions have been recently characterized by Karamardian, Schaible and Crouzeix (1993) who proved the following result:[3]

THEOREM 1.
Let $F\colon D \to \mathbb{R}^n$ be a differentiable function and D an open and convex subset of $\mathbb{R}^n$. Then F is pseudomonotone if and only if the following two conditions hold

(A) for all $x \in D$ and $v \in \mathbb{R}^n$

$$v \cdot F(x) = 0 \qquad \text{implies} \qquad v \cdot \partial F(x) v \leq 0$$

where $\partial F(x)$ is the Jacobian matrix of F.

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[3] For the proof see Karamardian, Schaible and Crouzeix (1993), Th. 4.1.

(B) if $v \cdot F(x) = v \cdot \partial F(x) v = 0$ then there exists $t_0 > 0$ such that

$$v \cdot F(x + tv) \leq 0 \qquad \text{for all} \qquad 0 < t < t_0$$

Condition (A) requires that the Jacobian matrix, $\partial F(x)$, be negative semidefinite on the subspace orthogonal to $F(x)$. This condition can be specified equivalently in terms of the signs of the minors of the bordered matrix[4]

$$\begin{bmatrix} H(x) & F(x) \\ F(x)^\top & 0 \end{bmatrix}$$

where $H(x)$ is the symmetric matrix $\partial F(x) + \partial F(x)^\top$. By letting $g(t) = v \cdot F(x+tv)$ for $t \geq 0$, condition (B) can be interpreted as requiring that whenever $g(0) = g'(0) = 0$ then $t = 0$ must be a local maximum for g.

In this section we show that, for a class of functions including demand functions, pseudomonotonicity is characterized only by condition (A) of Theorem 1. This result will be used then to derive a differential characterization of the weak axioms.

As a preliminary result let us prove the following

LEMMA 3.
Let $a < 0 < b$ and $g\colon [a, b] \to \mathbb{R}$ be a differentiable function with $g(0) = g'(0) = 0$. If $g(t) > 0$ for all $t \in (0, b]$ then

$$g'(\bar{t}) \geq \frac{g(\bar{t})}{\bar{t}} \qquad \text{for some} \qquad 0 < \bar{t} < b \tag{8}$$

Proof. Let us assume that (8) does not hold, that is,

$$g'(t) < \frac{g(t)}{t} \qquad \text{for all} \qquad t \in (0, b) \tag{$*$}$$

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[4] On determinantal criteria for semi-definiteness of symmetric matrices on a linear subspace see, for example, Debreu (1952). Crouzeix and Ferland (1982) provides still further equivalent conditions.

Since $\lim_{t \to 0} g(t)/t = g'(0) = 0$ the function $g(t)/t$ is well defined and continuous in the closed interval $[0, b]$ by setting its value equal to zero for $t = 0$. Let $m > 0$ be the maximum of $g(t)/t$ on $[0, b]$ and t_0 a point at which the function attains its maximum, i.e.[5]

$$\frac{g(t_0)}{t_0} = m \tag{$**$}$$

By $(*)$ and the definition of maximum we have

$$g'(t) < m \qquad \text{for all} \quad t \in (0, t_0)$$

On the other hand, by the mean value theorem, there exists $\bar{t} \in (0, t_0)$ such that

$$g'(\bar{t}) = \frac{g(t_0)}{t_0} = m$$

but this contradicts $(**)$ since $\bar{t} \in (0, t_0)$.

Q.E.D.

An immediate implication of Lemma 3 is that, in any neighborhood of zero, the ratio $g'(t)/g(t)$ is unbounded from above. Indeed, for any $M > 0$, there exists $0 < \bar{t} < 1/M$ such that

$$\frac{g'(\bar{t})}{g(\bar{t})} \geq \frac{1}{\bar{t}} > M$$

This is the result we are going to use to prove the following

THEOREM 2.
Let D be an open and convex subset of $\mathbb{R}^n$ and $F\colon D \to \mathbb{R}^n$ a continuously differentiable function with $F(x) \neq 0$ for any $x \in D$. Then F is pseudomonotone if and only if condition (A) holds, i.e.

$$v \cdot F(x) = 0 \qquad \text{implies} \qquad v \cdot \partial F(x) v \leq 0$$

for all $x \in D$ and $v \in \mathbb{R}^n$.

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[5] Clearly, since $g(t) > 0$ for $t \in (0, b)$ it must be $m > 0$ and $t_0 > 0$.

Proof. The only if part of the proof is standard. Let $v \cdot F(x) = 0$ and define $y_t = x + tv$ for all $t > 0$ such that $y_t \in D$. Therefore,

$$(y_t - x) \cdot F(x) = tv \cdot F(x) = 0$$

and since F is pseudo-monotone we also have

$$(y_t - x) \cdot F(y_t) = tv \cdot F(y_t) \leq 0$$

Hence

$$v \cdot \partial F(x)v = v \cdot \lim_{t \to 0} \frac{1}{t}[F(x + tv) - F(x)] = \lim_{t \to 0} \frac{1}{t} v \cdot F(y_t) \leq 0.$$

To show the converse let us suppose that condition (A) holds but F is not pseudomonotone, hence, by Lemma 1, there exist x and y in D such that

$$(y - x) \cdot F(x) = 0 \qquad \text{and} \qquad (y - x) \cdot F(y) > 0 \qquad (*)$$

Let $v = y - x$ and $y_t = x + tv$ for $0 \leq t \leq 1$ and define the function $g\colon [0,1] \to \mathbb{R}$ by

$$g(t) = v \cdot F(y_t)$$

By the assumptions on F, g is continuously differentiable and $g'(t) = v \cdot \partial F(y_t)v$. Moreover, from $(*)$, we have $g(0) = 0$, $g(1) > 0$ and by condition (A)

$$g'(0) = v \cdot \partial F(x)v \leq 0$$

Without loss of generality we also assume[6] that $g(t) > 0$ for all $0 < t < 1$, which in turns implies $g'(0) = 0$.

Now we find a violation of condition (A). Since $F(x) \neq 0$, for t sufficiently close to zero $F(x) \cdot F(y_t) > 0$ and the vector

$$u_t = v - \alpha(t)F(x)$$

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[6] Indeed, the set $B = \{t \in [0,1]\,;\, g(t) \leq 0\}$ is non empty and compact and has a maximum, $t^* < 1$. Therefore, set $x^* = x + t^*v$ and repeat the above argument with x^* and y.

is well defined with

$$\alpha(t) = \frac{g(t)}{F(x) \cdot F(y_t)}$$

By construction u_t and $F(y_t)$ are orthogonal and

$$\begin{aligned} u_t \cdot \partial F(y_t)u_t = {} & v \cdot \partial F(y_t)v + \alpha(t)^2 F(x) \cdot \partial F(y_t)F(x) \\ & - \alpha(t)v \cdot [\partial F(y_t) + \partial F(y_t)^T]F(x) \end{aligned}$$

This expression can be written as

$$u_t \cdot \partial F(y_t)u_t = g(t)\left[\frac{g'(t)}{g(t)} + \phi(t)\right] \qquad (**)$$

where

$$\phi(t) = \frac{1}{F(x) \cdot F(y_t)}[\alpha(t)F(x) \cdot \partial F(y_t)F(x) - \alpha(t)v \cdot [\partial F(y_t) + \partial F(y_t)^T]F(x)$$

Since $\phi(t)$ is continuous in $t = 0$ it is also bounded in a sufficiently small neighborhood of 0. On the other hand, since $g(t) > 0$, by Lemma 3 we know that $g'(t)/g(t)$ is unbounded from above near 0, so that the term in square brakets of $(**)$ can be made strictly positive. Therefore, we have a violation of condition (A) since we have shown that there exists a $\bar{t}$ such that $u_{\bar{t}} \cdot F(y_{\bar{t}}) = 0$ and $u_{\bar{t}} \cdot \partial F(y_{\bar{t}})u_{\bar{t}} > 0$.

Q.E.D.

Remark 1. The requirement that the function does not vanish in its domain is crucial for the result. If we drop this assumption, condition (A) is still necessary for pseudomonotonicity of F but it is no longer sufficient. For example, the one-variable function $F(x) = x^3$, with domain the real line, satisfies condition (A), but clearly it is not pseudomonotone since it is strictly increasing. The argument used in Theorem 2, however, can be used to refine Theorem 1. Indeed, let us introduce the following condition which is a weakening of condition (B) of Theorem 1:

(B') if $F(x) = 0$ then for any $t > 0$ there exists $0 < t_0 < t$ such that

$$v \cdot F(x + t_0 v) \leq 0$$

Unlike (B), the above condition need not be checked for any x in the domain of F but only at zeros, therefore (B') would be a more convenient condition than (B). Theorem 1 can be improved as follows.

THEOREM 1'
Let F be a continuously differentiable function from an open and convex subset of $\mathbb{R}^n$ to $\mathbb{R}^n$. Then F is pseudomonotone if and only if condition (A) and (B') hold.

Proof. The 'only if' part is immediate. To show the converse let us assume that F is not pseudomonotone and thus, as in the proof of Theorem 2 , we have $g(0) = v \cdot F(x) = 0$ and $g(t) > 0$ for all $0 < t < 1$. Then by (B') it must be $F(x) \neq 0$ and using the same argument as in Theorem 2 we obtain a contradiction.

Q.E.D.

This refinement of Theorem 1 was obtained by Crouzeix and Ferland (1996). Their proof, which, in essence, amounts to the proof of Theorem 2, follows a completely different line of argument and is rather lengthy.

Remark 2. The notions of generalized monotonicity are related to the notions of generalized concavity. For example, it is well known that the concavity of a real valued function, $h: D \to \mathbb{R}$ with $D \subset \mathbb{R}^n$, is characterized by the monotonicity of its gradient function, $\nabla h: D \to \mathbb{R}^n$. Similarly, the notion of pseudoconcavity of h, which is defined (in the differentiable case) by

$$(y - x) \cdot \nabla h(x) \leq 0 \qquad \text{implies} \qquad h(y) \leq h(x)$$

for any x and y in D, is characterized by pseudomonotonicity of its gradient ∇h.[7] For functions with non vanishing gradient, i.e. with $\nabla h(x) \neq 0$ for any $x \in D$, we obtain, as a special case of Theorem 2, the following second-order characterization of pseudoconcavity: for any x in D and v in $\mathbb{R}^n$

$$v \cdot \nabla h(x) = 0 \qquad \text{implies} \qquad v \cdot \partial^2 h(x) v \leq 0$$

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[7] This result was shown by Karamardian (1976). On the relationships between generalized monotonicity and generalized concavity see, among others, Karamardian and Schaible (1990).

where $\partial^2 h(x)$ is the Hessian Matrix. The above condition, which has been known since the early work of Arrow and Enthowen (1961) and Katzner (1970), also characterizes the weaker notion of quasiconcavity in the case $\nabla h(x) \neq 0$ for all $x \in D$.[8] Therefore, for functions with non vanishing gradient, quasiconcavity and pseudoconcavity are equivalent properties.

4. A differential characterization of the weak axioms

The results so far obtained are sufficient to derive a straightforward first-order characterization of the weak axioms. As for Wald's Axiom, combining Proposition 1 and Theorem 2 yields

PROPOSITION 3.
Let f be a continuously differentiable demand function. Then f satisfies Wald's Axiom if and only if

$$v \cdot f(p,w) = 0 \qquad \text{implies} \qquad v \cdot \partial_p f(p,w) v \leq 0$$

For the case of the Weak Axiom we derive a characterization involving not only the Jacobian matrix of demand but also the Slutsky matrix. Recall that the Slutsky matrix at (p,w), denoted by $S(p,w)$, is the Jacobian matrix of the compensated demand, $q \mapsto f(q, q \cdot f(p,w))$, evaluated at $q = p$, and is given by

$$S(p,w) = \partial_p f(p,w) + \partial_w f(p,w)\, f(p,w)^\top \tag{9}$$

We also recall that, by budget identity,

$$p\, S(p,w) = 0 \tag{10}$$

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[8] On characterizations of generalized concavity see Diewert, Avriel and Zang (1981), Crouzeix and Ferland (1982) and the references therein. In particular, on quasiconcavity see Crouzeix (1980) and Otani (1983).

PROPOSITION 4.
Let f be a continuously differentiable demand function. The following statements are equivalent.

a) f satisfies the Weak Axiom

b) the Slutsky matrix is negative semidefinite, i.e.

$$v \cdot S(p,w)v \leq 0 \qquad \text{for all} \qquad v \in \mathbb{R}^n$$

c) f is homogeneous and for all $v \in \mathbb{R}^n$

$$v \cdot f(p,w) = 0 \qquad \text{implies} \qquad v \cdot \partial_p f(p,w)v \leq 0$$

Proof. a) $\Rightarrow$ b). Let $v \in \mathbb{R}^n$ and set $q = p + tv$ for any $t \geq 0$ and sufficiently close to zero. By (a) and Proposition 2 we have that CLD holds, therefore

$$v \cdot [f(q, q \cdot f(p,w)) - f(p,w)] \leq 0$$

for all t. From the definition of directional derivative of compensated demand and differentiability we obtain

$$\lim_{t \to 0} \frac{1}{t}[f(q, q \cdot f(p,w)) - f(p,w)] = S(p,w)\, v$$

Combining the above results yields

$$0 \geq \lim_{t \to 0} \frac{1}{t} v \cdot [f(q, q \cdot f(p,w)) - f(p,w)] = v \cdot S(p,w)\, v$$

therefore the Slutsky matrix is negative semidefinite.

b) $\Rightarrow$ c). From (9) and b) it follows trivially that the Jacobian matrix is negative semidefinite on the orthogonal space to $f(p,w)$. To show that f is homogeneous it is sufficient to prove that $S(p,w)\,p = 0$. Let us Consider the symmetric matrix

$$\bar{S}(p,w) = S(p,w) + S(p,w)^{\top}$$

By b) this matrix is negative semidefinite and by (10)

$$p \cdot \bar{S}(p,w)\,p = 0$$

therefore by a standard result on quadratic forms we have $\bar{S}(p,w)p = 0$, hence

$$0 = \bar{S}(p,w)p = S(p,w)p + S(p,w)^{\top} p = S(p,w)p$$

and c) implies homogeneity of f.

Finally the implication from c) to a) follows readily from Theorem 2 and Proposition 2.

Q.E.D.

Propositions 1 to 4 contain the main results on the characterization of the weak axioms. At this stage it would be useful to highlight the connections of these results with the existing literature.

For the case of homogeneous demand, the equivalence between the Weak Axiom and negative semidefiniteness of the Slutsky matrix was first obtained by Kihlstrom, Mas-Colell and Sonnenschein (1976). Subsequently, Hildenbrand and Jerison (1989) proved the equivalence with condition (A) on the Jacobian matrix of demand. John (1995) clarified the role played by homogeneity of demand in the above results and was the first to show that this property is implied by the Weak Axiom.

Proposition 3 provides the same result as Theorem 1 in John (1995). John's proof is based on the introduction of a modified Slutsky matrix and follows a similar argument as in Hildenbrand and Jerison (1989). Our proof is more direct.

In Theorem 2, John (1995) shows the equivalence between the Weak Axiom and Wald's Axiom plus homogeneity. The same result follows from Proposition 1 and 2. We also provide an alternative proof that the Weak Axiom implies homogeneity (Lemma 2).

The results of Propositions 3 and 4 are also in Hildenbrand (1994). Hildenbrand's proof of the characterization of Wald's Axiom (which is mainly due to R. John) is very similar to the proof of our Theorem 2 with the exception of the result of Lemma 3. As for the characterization of the Weak Axiom, our proof of Proposition 4 is considerably shorter than Hildenbrand's because we use the first-order characterization of pseudomonotonicity.

5. Concluding remarks

Using the weak axioms in theoretical and applied work starting from the basic definitions is not always an easy task. There are cases where the differential characterization of the axioms helps to formulate sensible economic hypotheses suitable for subsequent empirical investigation. One important example is the analysis of market demand developed by Hildenbrand (1994).

Wald's Axiom or equivalently pseudomonotonicity is a desireable property of market demand since it ensures the convexity of the equilibrium set. If an equilibrium exists and the economy is regular then equilibrium is unique. In general, pseudomonotonicity cannot be expected to hold for market demand even when individual demands satisfy the Weak Axiom (or even when agents maximize preferences). Indeed, this property is not additive and therefore is not preserved under aggregation. However, in the case of large populations, Hildenbrand finds interesting economic conditions under which market demand acquires important structural properties and in particular satisfies Wald's Axiom. His approach combines the first-order characterization of Wald's Axiom with the Slutsky decomposition of the Jacobian matrix of individual demand into substitution and income effects.

Hildenbrand's model refers to a large population where each individual is described by a demand function, f, and a level of nominal income, w, i.e. by a pair of individual characteristics, (f, w). Income is supposed independent of prices and the economy is described by a distribution, μ, on the space of individual characteristics, $\mathcal{C} = \mathcal{F} \times \mathbb{R}_+$, where $\mathcal{F}$ is a set of admissible demand functions. For any given price vector p, μ induces a distribution $\nu(p)$ on the commodity space, that is the distribution of vectors of individual demand at price p. The mean of this distribution is market demand

$$F(p) = \int_{\mathcal{C}} f(p, w)\, d\mu$$

Using the Slutsky equation and assuming that all the integrals are well defined we can write the Jacobian matrix of market demand as

$$\partial F(p) = \bar{S}(p) - \bar{A}(p) \tag{11}$$

where $\bar{S}(p)$ is the average of the substitution matrices and $\bar{A}(p)$ is the average of income effect matrices, i.e.

$$\bar{A}(p) = \int_{\mathcal{C}} \partial_w f(p, w)\, f(p, w)^{\top}\, d\mu$$

Assuming that the Weak Axiom holds for individual demand functions we obtain, by Proposition 4, that the average substitution effect matrix, $\bar{S}(p)$, is negative semidefinite. Therefore, by (11) and Proposition 3, we have that market demand satisfies Wald's Axiom if the average income effect matrix $\bar{A}(p)$ is positive semidefinite on the hyperplane orthogonal to $F(p)$, that is, if

$$v \cdot F(p) = 0 \qquad \text{implies} \qquad v \cdot \bar{A}(p) v \geq 0 \tag{12}$$

To interpret condition (12) let us consider the distribution of demand vectors that would result if all incomes were shifted by the same small amount $\varepsilon > 0$, that is the distribution of $f(p, w + \varepsilon)$. This distribution is denoted by $\nu(p, \varepsilon)$ and $\operatorname{cov} \nu(p, \varepsilon)$ is its variance-covariance matrix; notice also that $\nu(p, 0) = \nu(p)$ and $\operatorname{cov} \nu(p, 0) = \operatorname{cov} \nu(p)$. We say that the distribution $\nu(p, \varepsilon)$ is 'more dispersed' around the mean than $\nu(p)$ in the direction $v \in \mathbb{R}^n$ if

$$v \cdot [\operatorname{cov} \nu(p, \varepsilon) - \operatorname{cov} \nu(p)] v \geq 0 \tag{13}$$

Hildenbrand (1994) shows that if $\nu(p, \varepsilon)$ tends to be 'more dispersed' than the actual distribution of demand vectors $\nu(p)$, for any direction orthogonal to $F(p)$, then condition (12) and thus Wald's Axiom for market demand hold.

To see how this hypothesis of 'increasing dispersion' of demand vectors leads to the result let us set

$$\partial_w \operatorname{cov} \nu(p) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\operatorname{cov} \nu(p, \varepsilon) - \operatorname{cov} \nu(p)]$$

By (13), the hypothesis of 'increasing dispersion' of demand vectors requires that

$$v \cdot \partial_w \operatorname{cov} \nu(p) v \geq 0 \qquad \text{for all} \qquad v \cdot F(p) = 0 \tag{14}$$

By using the definition of covariance between $\partial_w f_i(p, w)$ and $f_j(p, w)$ and the definition of $\bar{A}(p)$ we have that

$$\bar{A}(p)_{ij} = \operatorname{cov}[\partial_w f_i(p, w), f_j(p, w)] + \left[\int_{\mathcal{C}} \partial_w f_i(p, w) d\mu \right] F_j(p)$$

therefore, if $v \cdot F(p) = 0$ we obtain

$$v \cdot \bar{A}(p)v = \sum_{i,j} v_i v_j \mathrm{cov}\left[\partial_w f_i(p,w), f_j(p,w)\right]$$

On the other hand it is not difficult to see that

$$v \cdot \partial_w \mathrm{cov}\, \nu(p) v = 2 \sum_{i,j} v_i v_j \mathrm{cov}\left[\partial_w f_i(p,w), f_j(p,w)\right]$$

Therefore, by (14) we have

$$v \cdot \bar{A}(p)v = \frac{1}{2} v \cdot \partial_w \mathrm{cov}\, \nu(p) v \geq 0 \qquad \text{for all} \qquad v \cdot F(p) = 0$$

and condition (12) is satisfied.

The hypothesis of 'increasing dispersion' of demand vectors is justified for populations displaying sufficient etherogeneity in income and consumption behaviour. Under additional assumptions, this hypothesis has been tested using cross-section data taken from family expenditure surveys and seems to be well supported by empirical evidence.[9]

REFERENCES

Arrow, K. J. and A. C. Enthoven (1961) "Quasi-Concave Programming", *Econometrica*, 29, 779-800.

Crouzeix, J.P. (1980) "On Second Order Conditions for Quasiconvexity", *Mathematical Programming*, 18, 349-352.

Crouzeix, J.P. and J.A. Ferland (1982) "Criteria for Quasi-Convexity and Pseudo-Convexity: Relationships and Comparisons", *Mathematical Programming*, 23, 193-205.

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[9] See Hildenbrand (1994), Ch. 3.

Crouzeix, J.P. and J.A. Ferland (1996) "Criteria for Differentiable Generalized Monotone Maps", *Mathematical Programming*, 75, 399-406.

Debreu, G. (1952) "Definite and Semidefinite Quadratic Forms", *Econometrica*, 20, 295-300.

Diewert, W. E., Avriel, M. and I. Zang (1981) "Nine Kinds of Quasiconcavity and Concavity", *Journal of Economic Theory*, 25, 397-420.

Hildenbrand, W. (1994) *Market Demand: Theory and Empirical Evidence.* Princeton, NJ: Princeton University Press.

Hildenbrand, W. and M. Jerison (1989) "The Demand Theory of the Weak Axioms of Revealed Preferences", *Economics Letters*, 29, 209-213.

John, R. (1995) "The Weak Axiom of Revealed Preference and Homogeneity of Demand Functions", *Economics Letters*, 47, 11-16.

Karamardian, S. (1976) "Complementary Problems over Cones with Monotone and Pseudomonotone Maps", *Journal of Optimization Theory and Applications*, 18(4), 445-454.

Karamardian, S. and S. Schaible (1990) "Seven Kinds of Monotone Maps", *Journal of Optimization Theory and Applications*, 66(1), 37-46.

Karamardian, S., S. Schaible and J.P. Crouzeix (1993) "Characterization of Generalized Monotone Maps", *Journal of Optimization Theory and Applications*, 76, 399-413.

Katzner, D. W. (1970) *Static Demand Theory.* New York, MacMillan.

Kihlstrom, R., Mas-Colell, A. and H. Sonnenschein (1976) "The Demand Theory of the Weak Axiom of Revealed Preference", *Econometrica*, 44, 971-978.

Otani, K. (1983) "A Characterization of Quasi-Concave Functions" *Journal of Economic Theory*, 31, 194-196.

Samuelson, P. A. (1938) "A Note on the Pure Theory of Consumer Behaviour"; *Economica*, 5, 61-71.

1. Maria Cristina Marcuzzo [1985] "Yoan Violet Robinson (1903-1983)", pp. 134

2. Sergio Lugaresi [1986] "Le imposte nelle teorie del sovrappiù", pp. 26

3. Massimo D'Angelillo e Leonardo Paggi [1986] "PCI e socialdemocrazie europee. Quale riformismo?", pp. 158

4. Gian Paolo Caselli e Gabriele Pastrello [1986] "Un suggerimento hobsoniano su terziario ed occupazione: il caso degli Stati Uniti 1960/1983", pp. 52

5. Paolo Bosi e Paolo Silvestri [1986] "La distribuzione per aree disciplinari dei fondi destinati ai Dipartimenti, Istituti e Centri dell'Università di Modena: una proposta di riforma", pp. 25

6. Marco Lippi [1986] "Aggregations and Dynamic in One-Equation Econometric Models", pp. 64

7. Paolo Silvestri [1986] "Le tasse scolastiche e universitarie nella Legge Finanziaria 1986", pp. 41

8. Mario Forni [1986] "Storie familiari e storie di proprietà. Itinerari sociali nell'agricoltura italiana del dopoguerra", pp. 165

9. Sergio Paba [1986] "Gruppi strategici e concentrazione nell'industria europea degli elettrodomestici bianchi", pp. 56

10. Nerio Naldi [1986] "L'efficienza marginale del capitale nel breve periodo", pp. 54

11. Fernando Vianello [1986] "Labour Theory of Value", pp. 31

12. Piero Ganugi [1986] "Risparmio forzato e politica monetaria negli economisti italiani tra le due guerre", pp. 40

13. Maria Cristina Marcuzzo e Annalisa Rosselli [1986] "The Theory of the Gold Standard and Ricardo's Standard Comodity", pp. 30

14. Giovanni Solinas [1986] "Mercati del lavoro locali e carriere di lavoro giovanili", pp. 66

15. Giovanni Bonifati [1986] "Saggio dell'interesse e domanda effettiva. Osservazioni sul cap. 17 della General Theory", pp. 42

16. Marina Murat [1986] "Betwin old and new classical macroeconomics: notes on Lejonhufvud's notion of full information equilibrium", pp. 20

17. Sebastiano Brusco e Giovanni Solinas [1986] "Mobilità occupazionale e disoccupazione in Emilia Romagna", pp. 48

18. Mario Forni [1986] "Aggregazione ed esogeneità", pp. 13

19. Sergio Lugaresi [1987] "Redistribuzione del reddito, consumi e occupazione", pp. 17

20. Fiorenzo Sperotto [1987] "L'immagine neopopulista di mercato debole nel primo dibattito sovietico sulla pianificazione", pp. 34

21. M. Cecilia Guerra [1987] "Benefici tributari nel regime misto per i dividendi proposto dalla commissione Sarcinelli: una nota critica", pp. 9

22. Leonardo Paggi [1987] "Contemporary Europe and Modern America: Theories of Modernity in Comparative Perspective", pp. 38

23. Fernando Vianello [1987] "A Critique of Professor Goodwin's 'Critique of Sraffa'", pp. 12

24. Fernando Vianello [1987] "Effective Demand and the Rate of Profits. Some Thoughts on Marx, Kalecki and Sraffa", pp. 41

25. Anna Maria Sala [1987] "Banche e territorio. Approccio ad un tema geografico-economico", pp. 40

26. Enzo Mingione e Giovanni Mottura [1987] "Fattori di trasformazione e nuovi profili sociali nell'agricoltura italiana: qualche elemento di discussione", pp. 36

27. Giovanna Procacci [1988] "The State and Social Control in Italy During the First World War", pp. 18

28. Massimo Matteuzzi e Annamaria Simonazzi [1988] "Il debito pubblico", pp. 62

29. Maria Cristina Marcuzzo (a cura di) [1988] "Richard F. Kahn. A discipline of Keynes", pp. 118

30. Paolo Bosi [1988] "MICROMOD. Un modello dell'economia italiana per la didattica della politica fiscale", pp. 34

31. Paolo Bosi [1988] "Indicatori della politica fiscale. Una rassegna e un confronto con l'aiuto di MICROMOD", pp. 25

32. Giovanna Procacci [1988] "Protesta popolare e agitazioni operaie in Italia 1915-1918", pp. 45

33. Margherita Russo [1988] "Distretto Industriale e servizi. Uno studio dei trasporti nella produzione e nella vendita delle piastrelle", pp. 157

34. Margherita Russo [1988] "The effect of technical change on skill requirements: an empirical analysis", pp. 28

35. Carlo Grillenzoni [1988] "Identification, estimations of multivariate transfer functions", pp. 33

36. Nerio Naldi [1988] "'Keynes' concept of capital", pp. 40

37. Andrea Ginzburg [1988] "locomotiva Italia?", pp. 30

38. Giovanni Mottura [1988] "La 'persistenza' secolare. Appunti su agricoltura contadina ed agricoltura familiare nelle società industriali", pp. 40

39. Giovanni Mottura [1988] "L'anticamera dell'esodo. I contadini italiani della 'restaurazione contrattuale' fascista alla riforma fondiaria", pp. 40

40. Leonardo Paggi [1988] "Americanismo e riformismo. La socialdemocrazia europea nell'economia mondiale aperta", pp. 120

41. Annamaria Simonazzi [1988] "Fenomeni di isteresi nella spiegazione degli alti tassi di interesse reale", pp. 44

42. Antonietta Bassetti [1989] "Analisi dell'andamento e della casualità della borsa valori", pp. 12

43. Giovanna Procacci [1989] "State coercion and worker solidarity in Italy (1915-1918): the moral and political content of social unrest", pp. 41

44. Carlo Alberto Magni [1989] "Reputazione e credibilità di una minaccia in un gioco bargaining", pp. 56

45. Giovanni Mottura [1989] "Agricoltura familiare e sistema agroalimentare in Italia", pp. 84

46. Mario Forni [1989] "Trend, Cycle and 'Fortuitous cancellation': a Note on a Paper by Nelson and Plosser", pp. 4

47. Paolo Bosi , Roberto Golinelli , Anna Stagni [1989] "Le origini del debito pubblico e il costo della stabilizzazione", pp. 26

48. Roberto Golinelli [1989] "Note sulla struttura e sull'impiego dei modelli macroeconometrici", pp. 21

49. Marco Lippi [1989] "A Shorte Note on Cointegration and Aggregation", pp. 11

50. Gian Paolo Caselli e Gabriele Pastrello [1989] "The Linkage between Tertiary and Industrial Sector in the Italian Economy: 1951-1988. From an External Dependence to an International One", pp. 40

51. Gabriele Pastrello [1989] "Francois quesnay: dal Tableau Zig-zag al Tableau Formule: una ricostruzione", pp. 48

52. Paolo Silvestri [1989] "Il bilancio dello stato", pp. 34

53. Tim Mason [1990] "Tre seminari di storia sociale contemporanea", pp. 26

54. Michele Lalla [1990] "The Aggregate Escape Rate Analysed throught the Queueing Model", pp. 23

55. Paolo Silvestri [1990] "Sull'autonomia finanziaria dell'università", pp. 11

56. Paola Bertolini, Enrico Giovannetti [1990] "Uno studio di 'filiera' nell'agroindustria. Il caso del Parmigiano Reggiano", pp. 164

57. Paolo Bosi, Roberto Golinelli, Anna Stagni [1990] "Effetti macroeconomici, settoriali e distributivi dell'armonizzazione dell'IVA", pp. 24

58. Michele Lalla [1990] "Modelling Employment Spells from Emilia Labour Force Data", pp. 18

59. Andrea Ginzburg [1990] "Politica Nazionale e commercio internazionale", pp. 22

60. Andrea Giommi [1990] "La probabilità individuale di risposta nel trattamento dei dati mancanti", pp. 13

61. Gian Paolo Caselli e Gabriele Pastrello [1990] "The service sector in planned economies. Past experiences and future prospectives", pp. 32

62. Giovanni Solinas [1990] "Competenze, grandi industrie e distretti industriali,. Il caso Magneti Marelli", pp. 23

63. Andrea Ginzburg [1990] "Debito pubblico, teorie monetarie e tradizione civica nell'Inghilterra del Settecento", pp. 30

64. Mario Forni [1990] "Incertezza, informazione e mercati assicurativi: una rassegna", pp. 37

65. Mario Forni [1990] "Misspecification in Dynamic Models", pp. 19

66. Gian Paolo Caselli e Gabriele Pastrello [1990] "Service Sector Growth in CPE's: An Unsolved Dilemma", pp. 28

67. Paola Bertolini [1990] "La situazione agro-alimentare nei paesi ad economia avanzata", pp. 20

68. Paola Bertolini [1990] "Sistema agro-alimentare in Emilia Romagna ed occupazione", pp. 65

69. Enrico Giovannetti [1990] "Efficienza ed innovazione: il modello "fondi e flussi" applicato ad una filiera agro-industriale", pp. 38

70. Margherita Russo [1990] "Cambiamento tecnico e distretto industriale: una verifica empirica", pp. 115

71. Margherita Russo [1990] "Distretti industriali in teoria e in pratica: una raccolta di saggi", pp. 119

72. Paolo Silvestri [1990] " La Legge Finanziaria. Voce dell'enciclopedia Europea Garzanti", pp. 8

73. Rita Paltrinieri [1990] "La popolazione italiana: problemi di oggi e di domani", pp. 57

74. Enrico Giovannetti [1990] "Illusioni ottiche negli andamenti delle Grandezze distributive: la scala mobile e l''appiattimento' delle retribuzioni in una ricerca", pp. 120

75. Enrico Giovannetti [1990] "Crisi e mercato del lavoro in un distretto industriale: il bacino delle ceramiche. Sez I", pp. 150

76. Enrico Giovannetti [1990] " Crisi e mercato del lavoro in un distretto industriale: il bacino delle ceramiche. Sez. II", pp. 145

78. Antonietta Bassetti e Costanza Torricelli [1990] "Una riqualificazione dell'approccio bargaining alla selezioni di portafoglio", pp. 4

77. Antonietta Bassetti e Costanza Torricelli [1990] "Il portafoglio ottimo come soluzione di un gioco bargaining", pp. 15

79. Mario Forni [1990] "Una nota sull'errore di aggregazione", pp. 6

80. Francesca Bergamini [1991] "Alcune considerazioni sulle soluzioni di un gioco bargaining", pp. 21

81. Michele Grillo e Michele Polo [1991] "Political Exchange and the allocation of surplus: a Model of Two-party competition", pp. 34

82. Gian Paolo Caselli e Gabriele Pastrello [1991] "The 1990 Polish Recession: a Case of Truncated Multiplier Process", pp. 26

83. Gian Paolo Caselli e Gabriele Pastrello [1991] "Polish firms: Pricate Vices Pubblis Virtues", pp. 20

84. Sebastiano Brusco e Sergio Paba [1991] "Connessioni, competenze e capacità concorrenziale nell'industria della Sardegna", pp. 25

85. Claudio Grimaldi, Rony Hamaui, Nicola Rossi [1991] "Non Marketable assets and hauseholds' Portfolio Choice: a Case of Study of Italy", pp. 38

86. Giulio Righi, Massimo Baldini, Alessandra Brambilla [1991] "Le misure degli effetti redistributivi delle imposte indirette: confronto tra modelli alternativi", pp. 47

87. Roberto Fanfani, Luca Lanini [1991] "Innovazione e servizi nello sviluppo della meccanizzazione agricola in Italia", pp. 35

88. Antonella Caiumi e Roberto Golinelli [1992] "Stima e applicazioni di un sistema di domanda Almost Ideal per l'economia italiana", pp. 34

89. Maria Cristina Marcuzzo [1992] "La relazione salari-occupazione tra rigidità reali e rigidità nominali", pp. 30

90. Mario Biagioli [1992] "Employee financial participation in enterprise results in Italy", pp. 50

91. Mario Biagioli [1992] "Wage structure, relative prices and international competitiveness", pp. 50

92. Paolo Silvestri e Giovanni Solinas [1993] "Abbandoni, esiti e carriera scolastica. Uno studio sugli studenti iscritti alla Facoltà di Economia e Commercio dell'Università di Modena nell'anno accademico 1990/1991", pp. 30

93. Gian Paolo Caselli e Luca Martinelli [1993] "Italian GPN growth 1890-1992: a unit root or segmented trend representatin?", pp. 30

94. Angela Politi [1993] "La rivoluzione fraintesa. I partigiani emiliani tra liberazione e guerra fredda, 1945-1955", pp. 55

95. Alberto Rinaldi [1993] "Lo sviluppo dell'industria metalmeccanica in provincia di Modena: 1945-1990", pp. 70

96. Paolo Emilio Mistrulli [1993] "Debito pubblico, intermediari finanziari e tassi d'interesse: il caso italiano", pp. 30

97. Barbara Pistoresi [1993] "Modelling disaggregate and aggregate labour demand equations. Cointegration analysis of a labour demand function for the Main Sectors of the Italian Economy: 1950-1990", pp. 45

98. Giovanni Bonifati [1993] "Progresso tecnico e accumulazione di conoscenza nella teoria neoclassica della crescita endogena. Una analisi critica del modello di Romer", pp. 50

99. Marcello D'Amato e Barbara Pistoresi [1994] "The relationship(s) among Wages, Prices, Unemployment and Productivity in Italy", pp. 30

100. Mario Forni [1994] "Consumption Volatility and Income Persistence in the Permanent Income Model", pp. 30

101. Barbara Pistoresi [1994] "Using a VECM to characterise the relative importance of permanent and transitority components", pp. 28

102. Gian Paolo Caselli and Gabriele Pastrello [1994] "Polish recovery form the slump to an old dilemma", pp. 20

103. Sergio Paba [1994] "Imprese visibili, accesso al mercato e organizzazione della produzione", pp. 20

104. Giovanni Bonifati [1994] "Progresso tecnico, investimenti e capacità produttiva", pp. 30

105. Giuseppe Marotta [1994] "Credit view and trade credit: evidence from Italy", pp. 20

106. Margherita Russo [1994] "Unit of investigation for local economic development policies", pp. 25

107. Luigi Brighi [1995] "Monotonicity and the demand theory of the weak axioms", pp. 20

108. Mario Forni e Lucrezia Reichlin [1995] "Modelling the impact of technological change across sectors and over time in manufactoring", pp. 25

109. Marcello D'Amato and Barbara Pistoresi [1995] "Modelling wage growth dynamics in Italy: 1960-1990", pp. 38

110. Massimo Baldini [1995] "INDIMOD. Un modello di microsimulazione per lo studio delle imposte indirette", pp. 37

111. Paolo Bosi [1995] "Regionalismo fiscale e autonomia tributaria: l'emersione di un modello di consenso", pp. 38

112. Massimo Baldini [1995] "Aggregation Factors and Aggregation Bias in Consumer Demand", pp. 33

113. Costanza Torricelli [1995] "The information in the term structure of interest rates. Can stocastic models help in resolving the puzzle?" pp. 25

114. Margherita Russo [1995] "Industrial complex, pôle de développement, distretto industriale. Alcune questioni sulle unità di indagine nell'analisi dello sviluppo." pp. 45

115. Angelika Moryson [1995] "50 Jahre Deutschland. 1945 - 1995" pp. 21

116. Paolo Bosi [1995] "Un punto di vista macroeconomico sulle caratteristiche di lungo periodo del nuovo sistema pensionistico italiano." pp. 32

117. Gian Paolo Caselli e Salvatore Curatolo [1995] "Esistono relazioni stimabili fra dimensione ed efficienza delle istituzioni e crescita produttiva? Un esercizio nello spirito di D.C. North." pp. 11

118. Mario Forni e Marco Lippi [1995] "Permanent income, heterogeneity and the error correction mechanism." pp. 21

119. Barbara Pistoresi [1995] "Co-movements and convergence in international output. A Dynamic Principal Components Analysis" pp. 14

120. Mario Forni e Lucrezia Reichlin [1995] "Dynamic common factors in large cross-section" pp. 17

121. Giuseppe Marotta [1995] "Il credito commerciale in Italia: una nota su alcuni aspetti strutturali e sulle implicazioni di politica monetaria" pp. 20

122. Giovanni Bonifati [1995] "Progresso tecnico, concorrenza e decisioni di investimento: una analisi delle determinanti di lungo periodo degli investimenti" pp. 25

123. Giovanni Bonifati [1995] "Cambiamento tecnico e crescita endogena: una valutazione critica delle ipotesi del modello di Romer" pp. 21

124. Barbara Pistoresi e Marcello D'Amato [1995] "La riservatezza del banchiere centrale è un bene o un male? ,Effetti dell'informazione incompleta sul benessere in un modello di politica monetaria." pp. 32

125. Barbara Pistoresi [1995] "Radici unitarie e persistenza: l'analisi univariata delle fluttuazioni economiche." pp. 33

126. Barbara Pistoresi e Marcello D'Amato [1995] "Co-movements in European real outputs" pp. 20

127. Antonio Ribba [1996] "Ciclo economico, modello lineare-stocastico, forma dello spettro delle variabili macroeconomiche" pp. 31

128. Carlo Alberto Magni [1996] "Repeatable and una tantum real options a dynamic programming approach" pp. 23

129. Carlo Alberto Magni [1996] "Opzioni reali d'investimento e interazione competitiva: programmazione dinamica stocastica in optimal stopping" pp. 26

130. Carlo Alberto Magni [1996] "Vaghezza e logica fuzzy nella valutazione di un'opzione reale" pp. 20

131. Giuseppe Marotta [1996] "Does trade credit redistribution thwart monetary policy? Evidence from Italy" pp. 20

132. Mauro Dell'Amico e Marco Trubian [1996] "Almost-optimal solution of large weighted equicut problems" pp. 30

133. Carlo Alberto Magni [1996] "Un esempio di investimento industriale con interazione competitiva e avversione al rischio" pp. 20

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